Mathematics

Algebra I

DETROIT
PUBLIC SCHOOLS
COMMUNITY DISTRICT

Letter to Families from the DPSCD Office of Mathematics

Dear DPSCD Families,

The Office of Mathematics is partnering with families to support Distance Learning while students are home. We empower you to utilize the resources provided to foster a deeper understanding of grade-level mathematics.

In this packet, you will find links to videos, links to online practice, and pencil-and-paper practice problems. The Table of Contents shows day-by-day lessons from April 14th to June 19th. We encourage you to take every advantage of the material in this packet.

Daily lesson guidance can be found in the table of contents below. Each day has been designed to provide you access to materials from Khan Academy and the academic packet. Each lesson has this structure:

Watch: Khan Academy (if internet access is available)	Practice: Khan Academy (if internet access is available)	Pencil & Paper Practice: Academic Packet
Watch and take notes on the lesson video on Khan Academy	Complete the practice exercises on Khan Academy	Complete the pencil and paper practice.

If one-on-one, live support is required, please feel free to call the **Homework Hotline** at **1-833-466-3978**. Please check the Homework Hotline page for operating hours. We have DPSCD mathematics teachers standing by and are ready to assist.

We appreciate your continued dedication, support and partnership with Detroit Public Schools Community District and with your assistance we can press forward with our priority: Outstanding Achievement. Be safe. Be well!

Deputy Executive Director of K-12 Mathematics

Important Links and Information

Clever

Students access Clever by visiting www.clever.com/in/dpscd.

What are my username and password for Clever?
Students access Clever using their DPSCD login credentials. Usernames and passwords follow this structure:

Username: studentID@thedps.org
Ex. If Aretha Franklin is a DPSCD student with a student ID of 018765 her username would be 018765@thedps.org.

Password:
First letter of first name in upper case
First letter of last name in lower case
2-digit month of birth
2-digit year of birth
01 (male) or 02 (female)
For example: If Aretha Franklin's birthday is March 25, 1998, her password and password would be Af039802.

Accessing Khan Academy

To access Khan Academy, visit www.clever.com/in/dpscd. Once logged into Clever, select the Khan Academy button:

Khan Academy

Accessing Your CPM eBook

Students can access their CPM eBook in two ways:

Option 1: Access the eBook through <u>Clever</u>

1. Visit www.clever.com/in/dpscd. Login using your DPSCD credentials above.
2. Click on the CPM icon:

Option 2: Visit http://open-ebooks.cpm.org/

1. Visit the website listed above.
2. Click "I agree"
3. Select the CPM Geometry eBook:

CC Algebra

Desmos Online Graphing Calculator

Access to a free online graphing and scientific calculator can be found at https://www.desmos.com/calculator.

Week	Date	Topic	Watch (10 minutes)	Online Practice	Pencil & Paper Practice (25 minutes)
Week of 04/13-04/17	Monday Day 1	Holiday	N/A	N/A	N/A
	Day 2	4.1.1 Writing Equations	Construct Linear Equations from a Context	Writing Linear Equations (Slope-Intercept) from a Context	Problems 1-5
	Day 3	4.1.1 Writing Equations	Write a Linear System Interpreting a Solution to a System	Write a System from a Context Interpret the Solution to a System	Problems 6-10
	Day 4	4.1.1 & 4.2.1 Solving Systems by Substitution	Solve a System (Substitution)	Solve by Substitution Review	Problems 1-4

	Day 5	4.1.1 & 4.2.1 Solving Systems by Substitution	Systems using Substitution	Solve by Substitution	Problems 5-8
Week of 4/20-4/24	Day 1	4.1.1 & 4.2.1 Solving Systems by Substitution	Solve a System with Substitution	Solve a System with Substitution	Problems 9-12
	Day 2	4.2.3 through 4.2.5 Solving Systems by Elimination	Solve a System (Elimination)	Combine Equations Elimination Strategies	Problems 1-4
	Day 3	4.2.3 through 4.2.5 Solving Systems by Elimination	Solving by Elimination with Manipulation Solve a System with Elimination	Solve Systems by Elimination Elimination Review (Article)	Problems 5-8

	Day 4	4.2.3 through 4.2.5 Solving Systems by Elimination	Solve a Linear System (Elimination) Equivalent Systems Non-Equivalent Systems	Systems of Equations (All Methods Practice) Equivalent & Non-Equivalent Systems	Problems 9-12
	Day 5	5.1.1 through 5.1.3 Introduction to Sequences	Intro to Sequences	Intro to Seq. Practice	Problems 1-3
Week of 4/27-05/01	Day 1	5.1.1 through 5.1.3 Introduction to Sequences	Intro to Sequences	Intro to Seq. Practice	Problems 4-5
	Day 2	5.2.1 through 5.2.3 Equations for Sequences	Equations for Sequences	Equations for Sequences Practice	Problems 1-8

	Day 3	5.2.1 through 5.2.3 Equations for Sequences	Equations for Sequences	Equations for Sequences Practice	Problems 9-14
	Day 4	5.2.1 through 5.2.3 Equations for Sequences	Equations for Sequences	Equations for Sequences Practice	Problems 15-17
	Day 5	5.2.1 through 5.2.3 Equations for Sequences	Equations for Sequences	Equations for Sequences Practice	Problems 18-20
Week of 05/04-05/08	Day 1	5.3.1 Patterns of Growth in Tables and Graphs	Linear vs. Exp	Linear vs. Exp	Problems 1-8
	Day 2	5.3.1 Patterns of Growth in Tables and Graphs	Linear vs. Exp (Context)	Linear vs. Exp	Problems 9-16
	Day 3	7.1.1 through 7.1.6 Exponential Functions	Exp Growth Functions Intro (Equation)	Exp Expressions Word Problems	Problems 1-4

			Exp Growth Word Problem	Writing Exp Expressions	
	Day 4	7.1.1 through 7.1.6 Exponential Functions	Exponential Decay	Writing Exp Decay Equations	Problems 5-7
	Day 5	7.2.1 Fractional Exponents	Fractional Exponents	Fractional Exponent Practice	Problems 1-6
Week of 05/11-05/15	Day 1	7.2.1 Fractional Exponents	Negative Fractional Exponents	Fractional Exponent Practice	Problems 7-14
	Day 2	7.2.2 through 7.2.3 Curve Fitting	Curve Fitting	Curve Fitting Practice	Problems 1-3
	Day 3	7.2.2 through 7.2.3 Curve Fitting	Curve Fitting	Curve Fitting Practice	Problems 4-5

	Day 4	8.1.1 through 8.1.4 Factoring Quadratics	Factoring Quadratics	Factoring Quadratics Practice	Problems 1-7
	Day 5	8.1.1 through 8.1.4 Factoring Quadratics	Factoring Quadratics	Factoring Quadratics Practice	Problems 8-14
Week of 05/18-05/22	Day 1	8.1.1 through 8.1.4 Factoring Quadratics	Factoring Quadratics	Factoring Quadratics Practice	Problems 15-20
	Day 2	8.1.5 Factoring Shortcuts	Factoring Shortcuts	Factoring Shortcuts Practice	Problems 1-5
	Day 3	8.1.5 Factoring Shortcuts	Factoring Shortcuts	Factoring Shortcuts Practice	Problems 6-9

	Day 4	8.1.5 Factoring Shortcuts	Factoring Shortcuts	Factoring Shortcuts Practice	Problems 10-18
	Day 5	8.1.5 Factoring Shortcuts	Factoring Shortcuts	Perfect Squares	Problems 19-23
Week of 05/25-05/29	Day 1	8.1.5 Factoring Shortcuts	Factoring Shortcuts	Factoring Shortcuts Practice	Problems 24-27
	Day 2	8.2.2 and 8.2.3 Using the Zero Product Property	Zero Product Property	Zero Product Property Practice	Problems 1-5
	Day 3	8.2.2 and 8.2.3 Using the Zero Product Property	Zero Product Property	Zero Product Property Practice	Problems 6-10

	Day 4	8.2.2 and 8.2.3 Using the Zero Product Property	Zero Product Property	Zero Product Property Practice	Problems 11-15
	Day 5	8.2.5 Graphing Form and Completing the Square	Completing the Square	Completing the Square Practice	Problems 1-5
Week of 06/01-06/05	Day 1	8.2.5 Graphing Form and Completing the Square	Completing the Square	Completing the Square Practice	Problems 6-10
	Day 2	9.1.2 and 9.1.3 Using the Quadratic Formula	Using the Quadratic Formula	Using the Quadratic Formula Practice	Problems 1-4
	Day 3	9.1.2 and 9.1.3 Using the Quadratic Formula	Using the Quadratic Formula	Using the Quadratic Formula Practice	Problems 5-8
	Day 4	9.1.2 and 9.1.3 Using the Quadratic Formula	Using the Quadratic Formula	Using the Quadratic Formula Practice	Problems 9-12

	Day 5	9.2.1 and 9.2.2 Solving One-Variable Inequalities	Solving One-Variable Inequalities Practice	Solving One-Variable Inequalities Practice	Problems 1-5
Week of 06/08-06/12	Day 1	9.2.1 and 9.2.2 Solving One-Variable Inequalities	Solving One-Variable Inequalities Practice	Solving One-Variable Inequalities Practice	Problems 6-10
	Day 2	9.2.1 and 9.2.2 Solving One-Variable Inequalities	Solving One-Variable Inequalities Practice	Solving One-Variable Inequalities Practice	Problems 11-15
	Day 3	9.3.1 and 9.3.2 Graphing Two-Variable Inequalities	Graphing two-variable inequalities	Graphs of inequalities practice	Problems 1-4

	Day 4	9.3.1 and 9.3.2 Graphing Two-Variable Inequalities	Graphing two-variable inequalities	Graphs of inequalities practice	Problems 5-10
	Day 5	9.3.1 and 9.3.2 Graphing Two-Variable Inequalities	Graphing two-variable inequalities	Graphs of inequalities practice	Problems 11-13
Week of 06/15-06/19	Day 1	9.3.1 and 9.3.2 Graphing Two-Variable Inequalities	Graphing two-variable inequalities	Graphs of inequalities practice	Problems 14-15
	Day 2	9.4.1 through 9.4.3 Systems of Inequalities	Intro to graphing systems of inequalities	Graphs of inequalities Practice	Problems 1-3

	Day 3	9.4.1 through 9.4.3 Systems of Inequalities	Intro to graphing systems of inequalities	Graphs of inequalities Practice	Problems 4-6
	Day 4	9.4.1 through 9.4.3 Systems of Inequalities	Intro to graphing systems of inequalities	Graphs of inequalities Practice	Problems 7-8
	Day 5	9.4.1 through 9.4.3 Systems of Inequalities	Intro to graphing systems of inequalities	Graphs of inequalities Practice	Problems 9-10

WRITING EQUATIONS 4.1.1

In this lesson, students translate written information, often modeling everyday situations, into algebraic symbols and linear equations. Students use "let" statements to specifically define the meaning of each of the variables they use in their equations.

For additional examples and more problems, see the Checkpoint 7A problems at the back of the textbook.

Example 1

The perimeter of a rectangle is 60 cm. The length is 4 times the width. Write one or more equations that model the relationships between the length and width.

Start by identifying what is unknown in the situation. Then define variables, using "let" statements, to represent the unknowns. When writing "let" statements, the units of measurement must also be identified. This is often done using parentheses, as shown in the "let" statements below. In this problem, length and width are unknown.

Let w represent the width (cm) of the rectangle, and let l represent the length (cm).

In this problem there are two variables. To be able to find unique solutions for these two variables, two unique equations need to be written.

From the first sentence and our knowledge about rectangles, the equation $P = 2l + 2w$ can be used to write the equation $60 = 2l + 2w$. From the sentence "the length is 4 times the width" we can write $l = 4w$.

A system of equations is two or more equations that use the same set of variables to represent a situation. The system of equations that represent the situation is:

Let w represent the width (cm) of the rectangle, and let l represent the length (cm).

$$l = 4w$$

$$2l + 2w = 60$$

Note that students who took a CPM middle school course may recall a method called the 5-D Process. This 5-D Process is not reviewed in this course, but it is perfectly acceptable for students to use it to help write and solve equations for word problems.

Using a 5-D table:

	Define		**Do**	**Decide**
	Width	Length	Perimeter	$P = 60$?
Trial 1:	10	4(10)	2(40) + 2(10) = 100	too big
Trial 2:	5	4(5)	2(20) + 2(5) = 50	too small
	l	$4(l)$	$2(4w) + 2w = 60$	

Example 2

Mike spent \$11.19 on a bag containing red and blue candies. The bag weighed 11 pounds. The red candy costs \$1.29 a pound and the blue candy costs \$0.79 a pound. How much red candy did Mike have?

Start by identifying the unknowns. The question in the problem is a good place to look because it often asks for something that is unknown. In this problem, the amount of red candy and the amount of blue candy are unknown.

Let r represent the amount of red candy (lb), and b represent the amount of blue candy (lb).

Note how the units of measurement were defined. If we stated "r = red candy" it would be very easy to get confused as to whether r represented the *weight* of the candy or the *cost* of the candy.

From the statement "the bag weighed 11 pounds" we can write $r + b = 11$. Note that in this equation the units are lb + lb = lb, which makes sense.

The cost of the red candy will be \$1.29/pound multiplied by its weight, or $1.29r$. Similarly, the cost of the blue candy will be $0.79b$. Thus $1.29r + 0.79b = 11.19$.

Let r represent the weight of the red candy (lb), and let b represent the weight of the blue candy (lb).

$$r + b = 11$$
$$1.29r + 0.79b = 11.19$$

Problems

Write an equation or a system of equations that models each situation. Do not solve your equations.

1. A rectangle is three times as long as it is wide. Its perimeter is 36 units. Find the length of each side.

2. A rectangle is twice as long as it is wide. Its area is 72 square units. Find the length of each side.

3. The sum of two consecutive odd integers is 76. What are the numbers?

4. Nancy started the year with $425 in the bank and is saving $25 a week. Seamus started the year with $875 and is spending $15 a week. When will they have the same amount of money in the bank?

5. Oliver earns $50 a day and $7.50 for each package he processes at Company A. His paycheck on his first day was $140. How many packages did he process?

6. Dustin has a collection of quarters and pennies. The total value is $4.65. There are 33 coins. How many quarters and pennies does he have?

7. A one-pound mixture of raisins and peanuts costs $7.50. The raisins cost $3.25 a pound and the peanuts cost $5.75 a pound. How much of each ingredient is in the mixture?

8. An adult ticket at an amusement park costs $24.95 and a child's ticket costs $15.95. A group of 10 people paid $186.50 to enter the park. How many were adults?

9. Katy weighs 105 pounds and is gaining 2 pounds a month. James weighs 175 pounds and is losing 3 pounds a month. When will they weigh the same amount?

10. Harper Middle School has 125 fewer students than Holmes Junior High. When the two schools are merged there are 809 students. How many students attend each school?

Answers (Other equivalent forms are possible.)

	One Variable Equation	System of Equations	Let Statement
1.	$2w + 2(3w) = 36$	$l = 3w$ $2w + 2l = 36$	Let l = length, w = width
2.	$w(2w) = 7$	$l = 2w$ $lw = 72$	Let l = length, w = width
3.	$m + (m + 2) = 76$	$m + n = 76$ $n = m + 2$	Let m = the first odd integer and n = the next consecutive odd integer
4.	$425 + 25x = 875 - 15x$	$y = 425 + 25x$ $y = 875 - 15x$	Let x = the number of weeks and y = the total money in the bank
5.	$50 + 7.5p = 140$		Let p = the number of packages Oliver processed
6.	$0.25q + 0.01(33 - q) = 4.65$	$q + p = 33$ $0.25q + 0.01p = 4.65$	Let q = number of quarters, p = number of pennies
7.	$3.25r + 5.75(1 - r) = 7.5$	$r + p = 1$ $3.25r + 5.75p = 7.5(1)$	Let r = weight of raisins and p = weight of peanuts
8.	$24.95a + 15.95(10 - a) = 186.5$	$a + c = 10$ $24.95a + 15.95c = 186.5$	Let a = number of adult tickets and c = number of child's tickets
9.	$105 + 2m = 175 - 3m$	$w = 105 + 2m$ $w = 175 - 3m$	Let m = the number of months and w = the weight of each person
10.	$x + (x - 125) = 809$	$x + y = 809$ $y = x - 125$	Let x = number of Holmes students and y = number of Harper students

SOLVING SYSTEMS BY SUBSTITUTION 4.1.1 and 4.2.1

A system of equations has two or more equations with two or more variables. In the previous course, students were introduced to solving a system by looking at the intersection point of the graphs. They also learned the algebraic **Equal Values Method** of finding solutions. Graphing and the Equal Values Method are convenient when both equations are in $y = mx + b$ form (or can easily be rewritten in $y = mx + b$ form).

The **Substitution Method** is used to change a two-variable system of equations to a one-variable equation. This method is useful when one of the variables is isolated in one of the equations in the system. Two substitutions must be made to find both the x- and y-values for a complete solution.

For additional information, see the Math Notes boxes in Lessons 4.1.1, 4.2.2, 4.2.3, and 4.2.5. For additional examples and more problems solving systems using multiple methods, see the Checkpoint 7A and 7B problems at the back of the textbook.

The equation $x + y = 9$ has infinite possibilities for solutions: $(10, -1)$, $(2, 7)$, $(0, 9)$, …, but if $y = 4$ then there is only one possible value for x. That value is easily seen when we replace (substitute for) y with 4 in the original equation: $x + 4 = 9$, so $x = 5$ when $y = 4$. Substitution and this observation are the basis for the following method to solve systems of equations.

Example 1 (Equal Values Method)

Solve: $\begin{aligned} y &= 2x \\ x + y &= 9 \end{aligned}$

When using the Equal Values Method, start by rewriting both equations in $y = mx + b$ form. In this case, the first equation is already in $y = mx + b$ form. The second equation can be rewritten by subtracting x from both sides:

$$\begin{array}{rcl} x + y &=& 9 \\ -x & & -x \\ y &=& 9 - x \end{array}$$

Since y represents equal values in both the first equation, $y = 2x$, and the second equation, $y = 9 - x$, you can write: $2x = 9 - x$.

Solving for x:

$$\begin{array}{rcl} 2x &=& 9 - x \\ +x & & +x \\ 3x &=& 9 \\ x &=& 3 \end{array}$$

Then, either equation can be used to find y.
For example, use the first equation, $y = 2x$, to find y:

Since $x = 3$, and $y = 2x$:

$y = 2(3)$

$y = 6$

Example continues on next page →

Example continued from previous page.

The solution to this system of equation is $x = 3$ and $y = 6$. That is, the values $x = 3$ and $y = 6$ make both of the original equations true. When graphing, the point (3, 6) is the intersection of the two lines.

Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements:

For $y = 2x$ at (3, 6),

$6 \stackrel{?}{=} 2(3)$

$6 = 6$ is a true statement.

For $x + y = 9$ at (3, 6),

$3 + 6 \stackrel{?}{=} 9$

$9 = 9$ is a true statement.

Example 2 (Substitution Method)

Solve: $y = 2x$

$x + y = 9$

The same system of equations in Example 1 can be solved using the Substitution Method. From the first equation, y is equivalent to $2x$. Therefore you can replace the y in the second equation with $2x$:

$x + y = 9$

Replace y with $2x$, and solve.

$x + (2x) = 9$

$3x = 9$

$x = 3$

Then, either equation can be used to find y. For example, use the first equation, $y = 2x$, to find y:

$y = 2x$

Since $x = 3$,

$y = 2(3)$

$y = 6$

The solution to this system of equation is $x = 3$ and $y = 6$. That is, the values $x = 3$ and $y = 6$ make both of the original equations true. When graphing, the point (3, 6) is the intersection of the two lines.

Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements (see Example 1).

Example 3 (Substitution Method)

Look for a convenient substitution when using the Substitution Method; look for a variable that is by itself on one side of the equation.

Solve: $4x + y = 8$
$x = 5 - y$

Since x is equivalent to $5 - y$, replace x in the first equation with $5 - y$. Solve as usual.

$$\begin{aligned} 4x + y &= 8 \\ 4(5 - y) + y &= 8 \\ 20 - 4y + y &= 8 \\ 20 - 3y &= 8 \\ -3y &= -12 \\ y &= 4 \end{aligned}$$

To find x, use either of the original two equations. In this case, we will use $x = 5 - y$.

$$x = 5 - y$$

Since $y = 4$,

$$\begin{aligned} x &= 5 - (4) \\ x &= 1 \end{aligned}$$

The solution is $(1, 4)$. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

Example 4

Not all systems have a solution. If the substitution results in an equation that is not true, then there is **no solution**. The graph of a system of two linear equations that has no solutions is two parallel lines; there is no point of intersection. See the Math Notes box in Lesson 4.2.5 for additional information.

Solve: $y = 7 - 3x$
$3x + y = 10$

Replace y in the second equation with $7 - 3x$.

The resulting equation is never true. There is no solution to this system of equations.

$$\begin{aligned} 3x + y &= 10 \\ 3x + (7 - 3x) &= 10 \\ 3x - 3x + 7 &= 10 \\ 7 &\neq 10 \end{aligned}$$

Example 5

There may also be an **infinite number of solutions**. This graph would appear as a single line for the two equations.

Solve: $y = 4 - 2x$
$-4x - 2y = -8$

Substitute $4 - 2x$ in the second equation for y.

This statement is always true. There are infinite solutions to this system of equations.

$$\begin{aligned} -4x - 2y &= -8 \\ -4x - 2(4 - 2x) &= -8 \\ -4x - 8 + 4x &= -8 \\ -8 &= -8 \end{aligned}$$

Example 6

Sometimes you need to solve one of the equations for x or y to use the Substitution Method.

Solve: $y - 2x = -7$
$3x - 4y = 8$

Solve the first equation for y to get $y = 2x - 7$.
Then substitute $2x - 7$ in the second equation for y.

$$\begin{aligned} 3x - 4(2x - 7) &= 8 \\ 3x - 8x + 28 &= 8 \\ -5x + 28 &= 8 \\ -5x &= -20 \\ x &= 4 \end{aligned}$$

Then find y.

$y = 2x - 7$
Since $x = 4$,
$y = 2(4) - 7$
$y = 1$

Check: $1 - 2(4) = -7$ and $3(4) - 4(1) = 8$.

The solution is (4, 1). This would be the intersection point of the two lines. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

Problems

1. $y=-3x$ $4x+y=2$	2. $y=7x-5$ $2x+y=13$	3. $x=-5y-4$ $x-4y=23$
4. $x+y=10$ $y=x-4$	5. $y=5-x$ $4x+2y=10$	6. $3x+5y=23$ $y=x+3$
7. $y=-x-2$ $2x+3y=-9$	8. $y=2x-3$ $-2x+y=1$	9. $x=\frac{1}{2}y+\frac{1}{2}$ $2x+y=-1$
10. $a=2b+4$ $b-2a=16$	11. $y=3-2x$ $4x+2y=6$	12. $y=x+1$ $x-y=1$

Answers

1. (2, –6)	2. (2, 9)	3. (11, –3)
4. (7, 3)	5. (0, 5)	6. (1, 4)
7. (3, –5)	8. No solution	9. (0, –1)
10. (–12, –8)	11. Infinite solutions	12. No solution

ELIMINATION METHOD 4.2.3 through 4.2.5

In previous work with systems of equations, one of the variables was usually alone on one side of one of the equations. In those situations, it is convenient to rewrite both equations in $y = mx + b$ form and use the Equal Values Method, or to substitute the expression for one variable into the other equation using the Substitution Method.

Another method for solving systems of equations is the **Elimination Method**. This method is particularly convenient when both equations are in standard form (that is, $ax + by = c$). To solve this type of system, we can rewrite the equations by focusing on the coefficients. (The coefficient is the number in front of the variable.)

See problem 4-56 in the textbook for an additional explanation of the Elimination Method.

For additional information, see the Math Notes boxes in Lessons 4.2.4 and 5.1.1. For additional examples and more problems solving systems using multiple methods, see the Checkpoint 7B materials in the back of the textbook.

Example 1

Solve: $x - y = 2$
$2x + y = 1$

Recall that you are permitted to add the same expression to both sides of an equation. Since $x - y$ is equivalent to 2 (from the first equation), you are permitted to add $x - y$ to one side of the second equation, and 2 to the other side. Then solve.

$$\begin{aligned} 2x + y &= 1 \\ +x - y &\quad +2 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

Note that this was an effective way to eliminate y and find x because $-y$ and y were opposite terms; $y + (-y) = 0$.

Now substitute the value of x in either of the original equations to find the y-value.

$$2x + y = 1$$
Since $x = 1$,
$$\begin{aligned} 2(1) + y &= 1 \\ 2 + y &= 1 \\ y &= -1 \end{aligned}$$

The solution is $(1, -1)$, since $x = 1$ and $y = -1$ make both of the original equations true. On a graph, the point of intersection of the two original lines is $(1, -1)$. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

See problem 4-56 in the textbook for an additional explanation of the Elimination Method.

Example 2

Solve: $3x+6y=24$

$$3x+y=-1$$

Notice that both equations contain a $3x$ term. We can rewrite $3x+y=-1$ by multiplying both sides by -1, resulting in $-3x+(-y)=1$. Now the two equations have terms that are opposites: $3x$ and $-3x$. This will be useful in the next step because $-3x + 3x = 0$.

Since $-3x + (-y)$ is equivalent to 1, we can add $-3x + (-y)$ to one side of the equation and add 1 to the other side.

$$\begin{aligned} 3x+6y &= 24 \\ -3x+(-y) &\quad +1 \\ 5y &= 25 \\ y &= 5 \end{aligned}$$

Notice how the two opposite terms, $3x$ and $-3x$, eliminated each other, allowing us to solve for y.

Then substitute the value of y into either of the original equations to find x.

$$\begin{aligned} 3x+6(5) &= 24 \\ 3x+30 &= 24 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

The solution is $(-2, 5)$. It makes both equations true, and names the point where the two lines intersect on a graph. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

A more detailed explanation of this method can be found in the following example.

Example 3

To use the Elimination Method, one of the terms in one of the equations needs to be opposite of the corresponding term in the other equation. One of the equations can be multiplied to make terms opposite. For example, in the system at right, there are no terms that are opposite. However, if the first equation is multiplied by -4, then the two equations will have $4x$ and $-4x$ as opposites. The first equation now looks like this: $-4(x + 3y = 7) \rightarrow -4x + (-12y) = -28$. When multiplying an equation, be sure to multiply all the terms on *both* sides of the equation. With the first equation rewritten, the system of equations now looks like this:

$$\begin{aligned} x+3y &= 7 \\ 4x-7y &= -10 \end{aligned}$$

$$\begin{aligned} -4x+(-12y) &= -28 \\ 4x-7y &= -10 \end{aligned}$$

Since $4x - 7y$ is equivalent to -10, they can be added to both sides of the first equation:

$$\begin{array}{rr} & -4x+(-12y)=-28 \\ + & 4x-7y \quad\quad -10 \\ \hline & -19y \quad = -38 \\ & y \quad = 2 \end{array}$$

Now any of the equations can be used to find x:

Since $4x-7y=-10$ and $y=2$,

$$\begin{aligned} 4x-7(2) &= -10 \\ 4x-14 &= -10 \\ 4x &= 4 \\ x &= 1 \end{aligned}$$

The solution to the system of equations is $(1, 2)$.

Example 4

If multiplying one equation by a number will not make it possible to eliminate a variable, multiply both equations by different numbers to get coefficients that are the same or opposites.

Solve: $8x-7y=5$
$3x-5y=9$

One possibility is to multiply the first equation by 3 and the second equation by –8. The resulting terms 24*x* and –24*x* will be opposites, setting up the Elimination Method.

$$3(8x-7y=5)\Rightarrow 24x-21y=15$$
$$-8(3x-5y=9)\Rightarrow -24x+40y=-72$$

The system of equations is now:

$$24x-21y=15$$
$$-24x+40y=-72$$

This system can be solved by adding equivalent expressions (from the second equation) to the first equation:

$$\begin{array}{rl} 24x-21y & =15 \\ +\quad -24x+40y & -72 \\ \hline 19y & =-57 \\ y & =-3 \end{array}$$

Then solving for *x*, the solution is (–2, –3).

For an additional example like this one (where both equations had to be multiplied to create opposite terms), see Example 2 in the Checkpoint 7B materials at the back of the textbook.

Example 5

The special cases of "no solution" and "infinite solutions" can also occur. See the Math Notes box in Lesson 4.2.5 for additional information.

Solve: $4x+2y=6$
$2x+y=3$

Multiplying the second equation by 2 produces $4x + 2y = 6$. The two equations are identical, so when graphed there would be one line with *infinite* solutions because the same ordered pairs are true for both equations.

Solve: $2x+y=3$
$4x+2y=8$

Multiplying the first equation by 2 produces $4x + 2y = 6$. There are no numbers that could make $4x + 2y$ equal to 6, and $4x + 2y$ equal to 8 at the same time. The lines are parallel and there is *no solution*, that is, no point of intersection.

SUMMARY OF METHODS TO SOLVE SYSTEMS

Method	This Method is Most Efficient When	Example
Equal Values	Both equations in y-form.	$y = x - 2$ $y = -2x + 1$
Substitution	One variable is alone on one side of one equation.	$y = -3x - 1$ $3x + 6y = 24$
Elimination: Add to eliminate one variable.	Equations in standard form with opposite coefficients.	$x + 2y = 21$ $3x - 2y = 7$
Elimination: Multiply one equation to eliminate one variable.	Equations in standard form. One equation can be multiplied to create opposite terms.	$x + 2y = 3$ $3x + 2y = 7$
Elimination: Multiply both equations to eliminate one variable.	When nothing else works. In this case you could multiply the first equation by 3 and the second equation by –2, then add to eliminate the opposite terms.	$2x - 5y = 3$ $3x + 2y = 7$

Problems

1. $2x + y = 6$, $-2x + y = 2$
2. $-4x + 5y = 0$, $-6x + 5y = -10$
3. $2x - 3y = -9$, $x + y = -2$
4. $y - x = 4$, $2y + x = 8$
5. $2x - y = 4$, $\frac{1}{2}x + y = 1$
6. $-4x + 6y = -20$, $2x - 3y = 10$
7. $6x - 2y = -16$, $4x + y = 1$
8. $6x - y = 4$, $6x + 3y = -16$
9. $2x - 2y = 5$, $2x - 3y = 3$
10. $y - 2x = 6$, $y - 2x = -4$
11. $4x - 4y = 14$, $2x - 4y = 8$
12. $3x + 2y = 12$, $5x - 3y = -37$

Answers

1. $(1, 4)$
2. $(5, 4)$
3. $(-3, 1)$
4. $(0, 4)$
5. $(2, 0)$
6. Infinite
7. $(-1, 5)$
8. $(-\frac{1}{6}, -5)$
9. $(4.5, 2)$
10. No solution
11. $(3, -\frac{1}{2})$
12. $(-2, 9)$

INTRODUCTION TO SEQUENCES 5.1.1 through 5.1.3

In Chapter 5, students investigate sequences by looking for patterns and rules. Initially in the chapter, students concentrate on arithmetic sequences (sequences generated by adding a constant to the previous term), and then later in the chapter (and in Chapter 7) they consider geometric sequences (sequences generated by multiplying the previous term by a constant).

In Lessons 5.1.1 through 5.1.3 students are introduced to the two types of sequences, arithmetic and geometric, and their graphs, in everyday situations.

For additional examples and explanations, see the next section of this *Parent Guide with Extra Practice*, "Equations for Sequences." For additional information, see the first half of the Math Notes box in Lesson 5.3.2.

Example 1

Peachy Orchard Developers are preparing land to create a large subdivision of single-family homes. They have already built 15 houses on the site. Peachy Orchard plans to build six new homes every month. Create a table of values that will show the number of houses in the Peachy Orchard subdivision over time. Write an equation relating the number of houses over time. Graph the sequence.

Since the subdivision initially has 15 homes, 15 is the number of homes at time $t = 0$. After one month, there will be six more, or 21 homes. After the second month, there will be 27 homes. After each month, we add six homes to the total number of homes in the subdivision. Because we are adding a constant amount after each time period, this is an **arithmetic sequence**.

n, the number of months	$t(n)$, the total number of homes
0	15
1	21
2	27
3	33
4	39

We can find the equation for this situation by noticing that this is a linear function: the growth is constant. All arithmetic sequences are linear.

One way to write the equation that models this situation is to notice that the slope (growth) = 6 homes/month, and the y-intercept (starting point) = 15. Then in $y = mx + b$ form, the equation is $y = 6x + 15$.

Another way to find the equation of a line, especially in situations more complex than this one, is to use two points on the line, calculate the slope (m) between the two points, and then solve for the y-intercept (as in Lesson 2.3.2). This method is shown in the following steps:

Example continues on next page →

Example continued from previous page.

Choose (1, 21) and (4, 39)

$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$

$m = \frac{39-21}{4-1}$

$m = \frac{18}{3}$

$m = 6$

$y = mx + b$

at $(x, y) = (1, 21)$ and $m = 6$,

$21 = 6(1) + b$

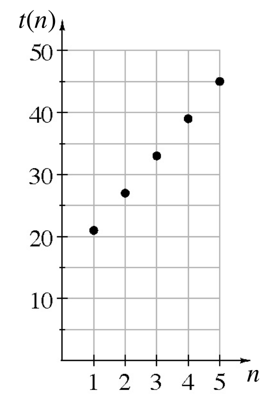
$b = 15$

$y = 6x + 15$

We write the equation as $t(n) = 6n + 15$ to show that this is an arithmetic sequence (as opposed to the linear function $y = mx + b$ or $f(x) = mx + b$) that will find the term t, for any number n. Let $t(n)$ represent the number of houses, and n the number of months.

The sequence would be written: 21, 27, 33, 39, …. Note that sequences usually begin with the first term (in this case, the term for the first month, $n = 1$).

The graph for the sequence is shown at right. Note that it is linear, and that it starts with the point (1, 21).



Example 2

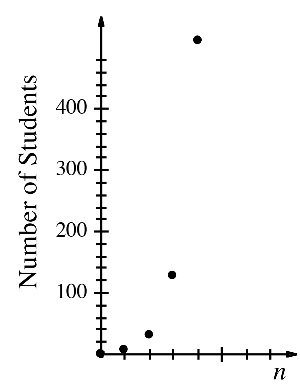
When Rosa tripped and fell into a muddy puddle at lunch (she was so embarrassed!), she knew exactly what would happen: within ten minutes, the two girls who saw her fall would each tell four people what they had seen. Within the next ten minutes, those eight students would each tell four more people. Rosa knew this would continue until everyone in the entire school was talking about her muddy experience. If there are 2016 students in the school, how many "generations" of gossiping would it take before everyone in the school was talking about Rosa? How many minutes would it take? Graph the situation.

At time $t = 0$, only two people see Rosa trip and fall. After ten minutes, each of those two people would tell four people; there are eight students gossiping about Rosa. After another ten minutes, each of those eight students will gossip with four more students; there will be $8 \times 4 = 32$ students gossiping. For the third increment of ten minutes, each of the 32 students will gossip with 4 students; $32 \times 4 = 128$ students gossiping.

Each time we multiply the previous number of students by four to get the next number of students. This is an example of a **geometric sequence**, and the multiplier is four. We record this in a table as shown at right, with n being the number of ten minute increments since Rosa's fall, and $t(n)$ is the number of students discussing the incident at that time. By continuing the table, we note that at time $t = 6$, there will be 2048 students discussing the mishap. Since there are only 2016 students in the school, everyone is gossiping by the sixth ten-minute increment of time. Therefore, just short of 60 minutes, or a little before one hour, everyone knows about Rosa's fall in the muddy puddle.

n, Number of Ten Minute Increments	Number of Students
0	2
1	8
2	32
3	128
4	512
5	1024
6	2048

The graph is shown at right. A geometric relationship is not linear; it is exponential. In future lessons, students would write the sequence as 8, 32, 128, … . Note that sequences usually begin with the first term (in this case, the term for the first month, $n = 1$).



Problems

1. Find the missing terms for this arithmetic sequence and an equation for $t(n)$.

$$\_\_, 15, 11, \_\_, 3$$

2. For this sequence each term is $\frac{1}{5}$ of the previous one. Work forward and backward to find the missing terms.

$$\text{—}, \text{—}, \frac{2}{3}, \text{—}, \text{—}$$

3. The 30th term of a sequence is 42. If each term in the sequence is four greater than the previous number, what is the first term?

4. The microscopic length of a crystalline structure grows so that each day it is 1.005 times as long as the previous day. If on the third day the structure was 12.5 nm long, write a sequence for how long it was on the first five days. (nm stands for nanometer, or 1×10^{-9} meters.)

5. Davis loves to ride the mini-cars at the amusement park but riders must be no more than 125 cm tall. If on his fourth birthday he is 94 cm tall and grows approximately 5.5 cm per year, at what age will he no longer be able to go on the mini-car ride?

Answers

1. 19 and 8; $t(n) = 23 - 4n$

2. $\frac{50}{3}, \frac{10}{3}, \frac{2}{3}, \frac{2}{15}, \frac{2}{75}$

3. $42 - 29(4) = -74$

4. $\approx 12.38, \approx 12.44, 12.5, \approx 12.56, \approx 12.63, \ldots$

5. $t(n) = 5.5n + 94$, so solve $5.5n + 94 \leq 125$. $n \approx 5.64$. At $\approx 4 + 5.64 = 9.64$ he will be too tall. Davis can continue to go on the ride until he is about $9\frac{1}{2}$ years old.

EQUATIONS FOR SEQUENCES 5.2.1 through 5.2.3

In these lessons, students learn multiple representations for sequences: as a string of numbers, as a table, as a graph, and as an equation. Read more about writing equations for sequences in the Math Notes box in Lesson 7.2.3.

In addition to the ways to write explicit equations for sequences, as explained in the Math Notes box in Lesson 7.2.3, equations for sequences can also be written recursively.
An explicit formula tells exactly how to find any specific term in the sequence. A recursive formula names the first term (or any other term) and how to get from one term to the next.
For an explanation of recursive sequences, see the Math Notes box in Lesson 5.3.2.

Example 1

This is the same scenario as in Example 1 of the previous section, *Introduction to Sequences*.

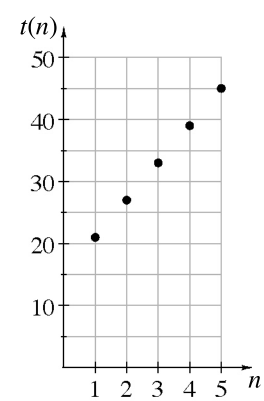
Peachy Orchard Developers are preparing land to create a large subdivision of single-family homes. They have already built 15 houses on the site. Peachy Orchard plans to build six new homes every month. Write a sequence for the number of houses built, then write an equation for the sequence. Fully describe a graph of this sequence.

The sequence is 21, 27, 33, 39, …. Note that sequences usually begin with the first term, where the number of months is $n = 1$.

The common difference is $m = 6$, and the zeroth term is $b = 15$. The equation can be written $t(n) = mn + b = 6n + 15$. Note that for a sequence, $t(n) =$ is used instead of $y =$.
$t(n) =$ indicates the equation is for a discrete sequence, as opposed to a continuous function. Students compared sequences to functions in Lesson 5.3.3.

The equation could also have been written as $a_n = 6n + 15$.

The graph of the sequence is shown at right. There are no x- or y-intercepts. There is no point at (0, 15) because sequences are usually written starting with the *first* term where $n = 1$. The domain consists of *integers* (whole numbers) greater than or equal to one. The range consists of the y-values of the points that follow the rule $t(n) = 6n + 15$ when $n \geq 1$. There are no asymptotes. The graph is linear and is shown at right. *This graph is discrete* (separate points). (Note: The related function, $y = 6x + 15$, would have the domain of all real numbers (including fractions and negatives) and the graph would be a continuous connected line.)



Example 2

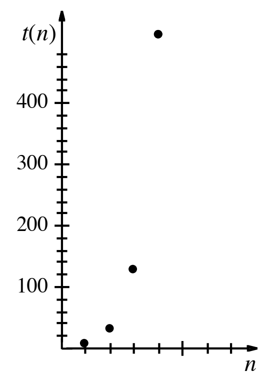
This is the same scenario as in Example 2 of the previous section, *Introduction to Sequences*.

When Rosa tripped and fell into a muddy puddle at lunch (she was so embarrassed!), she knew exactly what would happen: within ten minutes, the two girls who saw her fall would each tell four people what they had seen. Within the next ten minutes, those eight students would each tell four more people. Rosa knew this would continue until everyone in the entire school was talking about her muddy experience. Write a sequence for the number of people who knew about Rosa mishap in ten-minute intervals, then write an equation for the sequence. Fully describe a graph of this sequence.

The multiplier is $b = 4$, and the zeroth term is $a = 2$. The equation can be written $t(n) = ab^n = 2 \cdot 4^n$. The equation could also have been written as $a_n = 2 \cdot 4^n$. (Later, in Chapter 7, students will also learn "first term" notation for sequences, $a_n = 8 \cdot 4^{(n-1)}$.)

The sequence is: 8, 32, 128, 512, … . Note that the sequence is written starting with $n = 1$.

The graph of the sequence is to the right. There are no x- or y-intercepts. There is no point at (0, 2) because sequences are usually written starting with the *first* term where $n = 1$. The domain consists of *integers* (whole numbers) greater than or equal to one. The range consists of the y-values of the points that follow the rule $t(n) = 2(4)^n$ when $n \geq 1$. The graph is exponential and is shown at right. There is no symmetry. This graph is discrete (separate points). (Note: The related function, $y = 2 \cdot 4^n$ would have the domain of all real numbers (including fractions and negatives) and the graph would be a connected curve.)



Example 3

Consider the two sequences:

A: $-8,\ -5,\ -2,\ 1, \ldots$
B: $256,\ 128,\ 64, \ldots$

a. For each sequence, is it arithmetic, geometric, or neither? How can you tell? Explain completely.

b. What are the zeroth term and the generator for each sequence?

c. For each sequence, write an equation representing the sequence.

d. Is 378 a term of sequence A? Justify your answer.

e. Is $\frac{1}{4}$ a term of sequence B? Justify your answer.

To determine the type of sequence for A and B above, we have to look at the growth of each sequence.

A: $-8,\ -5,\ -2,\ 1,\ \ldots$
\ / \ / \ /
$+3\quad +3\quad +3$

Sequence A is made (generated) by adding three to each term to get the next term. When each term has a **common difference** (in this case, "+3") the sequence is **arithmetic**.

Sequence B, however, is different. The terms do not have a common difference.

B: $256,\ 128,\ 64,\ \ldots$
\ / \ /
$-128\quad -64$

Instead, these terms have a **common ratio** (multiplier). A sequence with a common ratio is a **geometric sequence**.

B: $256,\ 128,\ 64,\ \ldots$
\ / \ /
$\cdot\frac{1}{2}\quad \cdot\frac{1}{2}$

The first term for sequence A is –8, and has a generator or common difference of +3. Therefore the zeroth term is –11 (because $-11 + 3 = -8$). An arithmetic sequence has an equation of the form $t(n) = mn + b$ (or $a_n = mn + a_0$) where m is the common difference, and b is the initial value. For sequence A, the equation is $t(n) = 3n - 11$, for $n = 1, 2, 3, \ldots$

Example continues on next page →

Example continued from previous page.

For sequence B, the first term is 256 with a generator or common ratio of $\frac{1}{2}$. Therefore the zeroth term is 512, because $512 \cdot \frac{1}{2} = 256$. The general equation for a geometric sequence is $t(n) = ab^n$ where a is the zeroth term, and b is the common ratio (multiplier). For sequence B, the equation is $t(n) = 512\left(\frac{1}{2}\right)^n$ for $n = 1, 2, 3, \ldots$

To check if 378 is a term in sequence A, we could list the terms of the sequence out far enough to check, but that would be time consuming. Instead, we will check if there is an integer n that solves $t(n) = 3n - 11 = 378$.

$$\begin{aligned} 3n - 11 &= 378 \\ 3n &= 389 \\ n &= \tfrac{389}{3} = 129\tfrac{2}{3} \end{aligned}$$

When we solve, n is not a whole number, therefore 378 cannot be a term in the sequence.

Similarly, to check if $\frac{1}{4}$ is a term in sequence B, we need to solve $t(n) = 512\left(\frac{1}{2}\right)^n = \frac{1}{4}$, and look for a whole number solution.

$$\begin{aligned} 512\left(\tfrac{1}{2}\right)^n &= \tfrac{1}{4} \\ \tfrac{1}{512} \cdot 512\left(\tfrac{1}{2}\right)^n &= \tfrac{1}{512} \cdot \tfrac{1}{4} \\ \left(\tfrac{1}{2}\right)^n &= \tfrac{1}{2048} \\ \left(\tfrac{1}{2}\right)^n &= \tfrac{1}{2^{11}} \\ \tfrac{1}{2^n} &= \tfrac{1}{2^{11}} \\ n &= 11 \end{aligned}$$

Although solving an equation like this is probably new for most students, they can solve this problem by using guess-and-check. Also, by writing both sides as a power of 2, students can see the solution easily.

Since the equation has a whole number solution, $\frac{1}{4}$ is a term of sequence B.

That is, when $n = 11$, $t(n) = \frac{1}{4}$.

EXAMPLES FOR ARITHMETIC SEQUENCES

List the first five terms of each arithmetic sequence.

Example 4 (An explicit formula)

$$t(n) = 5n + 2$$

$t(1) = 5(1) + 2 = 7$
$t(2) = 5(2) + 2 = 12$
$t(3) = 5(3) + 2 = 17$
$t(4) = 5(4) + 2 = 22$
$t(5) = 5(5) + 2 = 27$

The sequence is: 7, 12, 17, 22, 27, …

Example 5 (A recursive formula)

$$t(1) = 3,\ t(n+1) = t(n) - 5$$

$t(1) = 3$
$t(2) = t(1) - 5 = 3 - 5 = -2$
$t(3) = t(2) - 5 = -2 - 5 = -7$
$t(4) = t(3) - 5 = -7 - 5 = -12$
$t(5) = t(4) - 5 = -12 - 5 = -17$

The sequence is: 3, –2, –7, –12, –17, …

Example 6

Find an explicit and a recursive formula for the sequence: $-2, 1, 4, 7, \ldots$

Explicit: $m = 3$, $b = -5$, so the equation is: $t(n) = mn + b = 3n - 5$

Recursive: $t(1) = -2$, $t(n + 1) = t(n) + 3$

EXAMPLES FOR GEOMETRIC SEQUENCES

List the first five terms of each geometric sequence.

Example 7 (An explicit formula)

$$t(n) = 3 \cdot 2^{n-1}$$

$t(1) = 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3$

$t(2) = 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6$

$t(3) = 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12$

$t(4) = 3 \cdot 2^{4-1} = 3 \cdot 2^3 = 24$

$t(5) = 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48$

The sequence is: $3, 6, 12, 24, 48, \ldots$

Example 8 (A recursive formula)

$$t(1) = 8,\ t(n+1) = t(n) \cdot \tfrac{1}{2}$$

$t(1) = 8$

$t(2) = t(1) \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$

$t(3) = t(2) \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$

$t(4) = t(3) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$

$t(5) = t(4) \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$

The sequence is: $8, 4, 2, 1, \frac{1}{2}, \ldots$

Example 9

Find an explicit and a recursive formula for the sequence: $81, 27, 9, 3, \ldots$

Explicit: $a_1 = 81$, $b = \frac{1}{3}$ so the a_0 (the zeroth term) is found by $a_0 = 81 \div \frac{1}{3} = 243$ and the equation is: $a_n = a_0 \cdot b^n = 243 \cdot \left(\frac{1}{3}\right)^n$, or alternatively, $t(n) = 243 \cdot \frac{1}{3}^n$

Recursive: $t(1) = 81,\ t(n+1) = t(n) \cdot \frac{1}{3}$

Problems

Each of the functions listed below defines a sequence. List the first five terms of the sequence, and state whether the sequence is arithmetic, geometric, both, or neither.

1. $t(n) = 5n + 2$
2. $s_n = 3 - 8n$
3. $u(n) = 9n - n^2$
4. $t(n) = (-4)^n$
5. $s(n) = \left(\frac{1}{4}\right)^n$
6. $u(n) = n(n + 1)$
7. $t(n) = 8$
8. $s_n = \frac{3}{4}n + 1$

Identify each of the following sequences as arithmetic or geometric. Then write the equation that gives the terms of the sequence.

9. $48, 24, 12, 6, 3, \ldots$
10. $-4, 3, 10, 17, 24, \ldots$
11. $43, 39, 35, 31, 27, \ldots$
12. $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \frac{9}{32}, \frac{27}{128}, \ldots$
13. $5, -5, 5, -5, 5, \ldots$
14. $10, 1, 0.1, 0.01, 0.001, \ldots$

Graph the following two sequences on the same set of axes.

15. $t(n) = -6n + 20$
16. $1, 4, 16, 64, \ldots$

17. Do the two sequences of the last two problems have any terms in common? Explain how you know.
18. Every year since 1548, the average height of a human male has increased slightly. The new height is 100.05% of what it was the previous year. If the average male's height was 54 inches in 1548, what was the average height of a male in 2008?
19. Davis has \$5.40 in his bank account on his fourth birthday. If his parents add \$0.40 to his bank account every week, when will he have enough to buy the new Smokin' Derby race car set which retails for \$24.99?
20. Fully describe the graph of the sequence $t(n) = -4n + 18$.

Arithmetic Sequences

List the first five terms of each arithmetic sequence.

21. $t(n) = 5n - 2$

22. $t(n) = -3n + 5$

23. $t(n) = -15 + \frac{1}{2}n$

24. $t(n) = 5 + 3(n - 1)$

25. $t(1) = 5, t(n + 1) = t(n) + 3$

26. $t(1) = 5, t(n + 1) = t(n) - 3$

27. $t(1) = -3, t(n + 1) = t(n) + 6$

28. $t(1) = \frac{1}{3}, t(a + 1) = t(n) + \frac{1}{2}$

Find the 30$^{\text{th}}$ term of each arithmetic sequence.

29. $t(n) = 5n - 2$

30. $t(n) = -15 + \frac{1}{2}n$

31. $t(31) = 53, d = 5$

32. $t(1) = 25, t(n + 1) = t(n) - 3$

For each arithmetic sequence, find an explicit and a recursive formula.

33. 4, 8, 12, 16, 20, …

34. –2, 5, 12, 19, 26, …

35. 27, 15, 3, –9, –21, …

36. $3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{3}, \ldots$

Sequences are graphed using points of the form: (term number, term value).
For example, the sequence 4, 9, 16, 25, 36, … would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), …. Sequences are graphed as points and not connected.

37. Graph the sequences from problems 21 and 22 above and determine the slope of each line.

38. How does the slope of the line found in the previous problem relate to the sequence?

Geometric Sequences

List the first five terms of each geometric sequence.

39. $t(n) = 5 \cdot 2^n$

40. $t(n) = -3 \cdot 3^n$

41. $t(n) = 40\left(\frac{1}{2}\right)^{n-1}$

42. $t(n) = 6\left(-\frac{1}{2}\right)^{n-1}$

43. $t(1) = 5,\ t(n+1) = t(n) \cdot 3$

44. $t(1) = 100,\ t(n+1) = t(n) \cdot \frac{1}{2}$

45. $t(1) = -3,\ t(n+1) = t(n) \cdot (-2)$

46. $t(1) = \frac{1}{3},\ t(n+1) = t(n) \cdot \frac{1}{2}$

Find the 15th term of each geometric sequence.

47. $t(14) = 232,\ r = 2$

48. $t(16) = 32,\ r = 2$

49. $t(14) = 9,\ r = \frac{2}{3}$

50. $t(16) = 9,\ r = \frac{2}{3}$

Find an explicit and a recursive formula for each geometric sequence.

51. 2, 10, 50, 250, 1250, …

52. $16, 4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$

53. 5, 15, 45, 135, 405, …

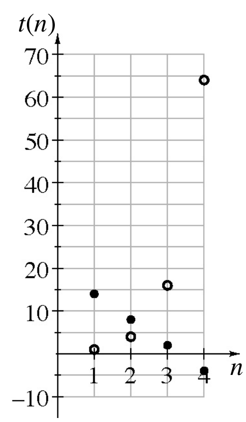
54. 3, –6, 12, –24, 48, …

55. Graph the sequences from problems 39 and 52. Remember the note before problem 37 about graphing sequences.

56. How are the graphs of geometric sequences different from arithmetic sequences?

Answers

1. 7, 12, 17, 22, 27, arithmetic, common difference is 5.
2. –5, –13, –21, –29, –37, arithmetic, common difference is –8.
3. 8, 14, 18, 20, 20, neither
4. –4, 16, –64, 256, –1024, geometric, common ratio is –4.
5. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$, geometric, common ratio is $\frac{1}{4}$.
6. 2, 6, 12, 20, 30, neither
7. 8, 8, 8, 8, 8, both, common difference is 0, common ratio is 1.
8. $\frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4, \frac{19}{4}$, arithmetic, common difference is $\frac{3}{4}$.
9. geometric, $t(n) = 96\left(\frac{1}{2}\right)^n$
10. arithmetic, $t(n) = 7n - 11$
11. arithmetic, $t(n) = -4n + 47$
12. geometric, $t(n) = \left(\frac{8}{9}\right)\left(\frac{3}{4}\right)^n$
13. geometric, $t(n) = -5(-1)^n$
14. geometric, $t(n) = 100\left(\frac{1}{10}\right)^n$
15. See dots on graph.
16. See circles on graph.



17. No, they do not. The graph is discrete, or just points, which are the terms of each sequence. Since they do not share a common point, they do not have any terms in common.
18. In 2008, $54(1.0005)^{460} \approx 67.96$ inches.
19. $t(n) = 0.4n + 5.4$, so solve $0.4n + 5.4 \geq 24.99$. $n = 48.975$. In 49 weeks he will have $25. If he must also cover tax, he will need another three or four weeks.
20. This is a function, it represents an arithmetic sequence, the graph is discrete but the points are linear. The domain is the positive integers: 1, 2, 3, …. The range is the sequence itself: 14, 10, 6, 2, –2, …. There are no asymptotes.

21. 3, 8, 13, 18, 23

22. 2, –1, –4, –7, –10

23. $-14\frac{1}{2}$, –14, $-13\frac{1}{2}$, –13, $-12\frac{1}{2}$

24. 5, 8, 11, 14, 17

25. 5, 8, 11, 14, 17

26. 5, 2, –1, –4, –7

27. –3, 3, 9, 15, 21

28. $\frac{1}{3}, \frac{5}{6}, 1\frac{1}{3}, 1\frac{5}{6}, 2\frac{1}{3}$

29. 148

30. 0

31. 48

32. –62

33. $t(n) = 4n$; $t(1) = 4$, $t(n+1) = t(n) + 4$

34. $t(n) = 7n - 9$; $t(1) = -2$, $t(n+1) = t(n) + 7$

35. $t(n) = 39 - 12n$; $t(1) = 27$, $t(n+1) = t(n) - 12$

36. $a_n = \frac{1}{3}n + 2\frac{2}{3}$; $a_1 = 3$, $a_{n+1} = a_n + \frac{1}{3}$

37. Graph (21): (1, 3), (2, 8), (3, 13), (4, 18), (5, 23) slope = 5

Graph (22): (1, 2), (2, –1), (3, –4), (4, –7), (5, –10) slope = –3

38. The slope of the line containing the points is the same as the common difference of the sequence.

39. 10, 20, 40, 80, 160

40. –9, –27, –81, –243, –729

41. 40, 20, 10, 5, $\frac{5}{2}$

42. 6, –3, $\frac{3}{2}, -\frac{3}{4}, \frac{3}{8}$

43. 5, 15, 45, 135, 405

44. 100, 50, 25, $\frac{25}{2}, \frac{25}{4}$

45. –3. 6, –12, 24, –48

46. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$

47. 464

48. 16

49. 6

50. $\frac{27}{2}$

51. $t(n) = \frac{2}{5} \cdot 5^n$; $t(1) = 2$, $t(n+1) = t(n) \cdot 5$

52. $t(n) = 64 \cdot \left(\frac{1}{4}\right)^n$; $t(1) = 16$, $t(n+1) = t(n) \cdot \frac{1}{4}$

53. $t(n) = \frac{5}{3} \cdot 3^n$; $t(1) = 5$, $t(n+1) = t(n) \cdot 3$

54. $t(n) = \frac{-3}{2} \cdot (-2)^n$; $t(1) = 3$, $t(n+1) = t(n) \cdot (-2)$

55. Graph (39): (1, 10), (2, 20), (3, 40), (4, 80), (5, 160)

Graph (52): (1, 16), (2, 4), (3, 1), (4, $\frac{1}{4}$), (5, $\frac{1}{16}$)

56. Arithmetic sequences are linear and geometric sequences are curved (exponential).

PATTERNS OF GROWTH IN TABLES AND GRAPHS 5.3.1

To recognize if a function is linear, exponential, or neither, look at the differences of the y-values between consecutive integer x-values. If the difference is constant, the graph is linear. If the difference is not constant, look at the pattern in the y-values. If a constant multiplier can be used to move from one y-value to the next, then the function is exponential. (Note that the same multiplier can be used to move from difference to difference in an exponential function.)

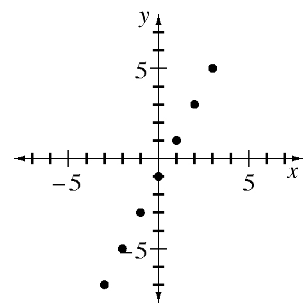
Examples

Based on each table, identify the shape of the graph.

Example 1

x	–3	–2	–1	0	1	2	3
y	–7	–5	–3	–1	1	3	5

2 2 2 2 2 2

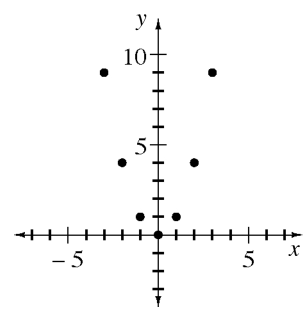


The difference in y-values is always two, a constant.
The function is linear; the graph at right confirms this.

Example 2

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

–5 –3 –1 1 3 5

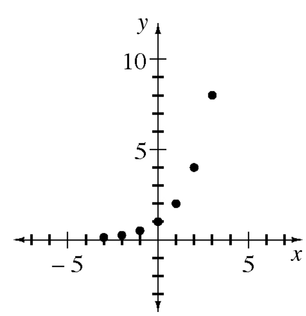


The first difference in y-values is not constant, and there is not a constant multiplier in moving from one y-value to the next. The function is neither linear nor exponential; the graph at right confirms this.

Example 3

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{2}$ 1 2 4



The y-values have a constant multiplier of 2. (Also the differences in y-values have a constant multiplier of 2.)
The function is exponential; the graph at right confirms this.

Problems

Based on the growth (the difference in y-values) shown in the tables, identify the corresponding graph as linear, exponential, or neither.

1.

x	–3	–2	–1	0	1	2	3
y	14	10	6	2	–2	–6	–10

2.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{2}$	1	2	4	8	16	32

3.

x	–3	–2	–1	0	1	2	3
y	21	12	5	0	–3	–4	–3

4.

x	–3	–2	–1	0	1	2	3
y	–16	–13	–10	–7	–4	–1	2

5.

x	–3	–2	–1	0	1	2	3
y	–14	–9	–4	1	6	11	16

6.

x	–3	–2	–1	0	1	2	3
y	–18	–6	–2	0	2	6	18

7.

x	–3	–2	–1	0	1	2	3
y	4	8	16	32	64	128	256

8.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

9.

x	–3	–2	–1	0	1	2	3
y	30	20	12	6	2	0	0

10.

x	–3	–2	–1	0	1	2	3
y	11	9	7	5	3	1	–1

11.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

12.

x	–3	–2	–1	0	1	2	3
y	–27	–9	–3	0	3	9	27

13.

x	–3	–2	–1	0	1	2	3
y	0	5	8	9	8	5	0

14.

x	–3	–2	–1	0	1	2	3
y	3	0	–1	0	3	8	15

15.

x	–3	–2	–1	0	1	2	3
y	1	0	–1	–2	–1	0	1

16.

x	–3	–2	–1	0	1	2	3
y	$\frac{9}{8}$	$\frac{9}{4}$	$\frac{9}{2}$	9	18	36	72

Answers

1.	linear	2.	exponential
3.	neither	4.	linear
5.	linear	6.	neither
7.	exponential	8.	exponential
9.	neither	10.	linear
11.	exponential	12.	neither
13.	neither	14.	neither
15.	neither	16.	exponential

EXPONENTIAL FUNCTIONS 7.1.1 through 7.1.6

In these sections, students generalize what they have learned about geometric sequences to investigate exponential functions. Students study exponential functions of the form $y = ab^x$. Students look at multiple representations of exponential functions, including graphs, tables, equations, and context. They learn how to move from one representation to another. Students learn that the value of a is the "starting value" of the function—a is the y-intercept or the value of the function at $x = 0$. b is the growth (multiplier). If $b > 1$ then the function increases; if b is a fraction between 0 and 1 (that is, $0 < b < 1$), then the function decreases (decays). In this course, values of $b < 0$ are not considered.

For additional information, see the Math Notes boxes in Lesson 7.1.3 and 7.2.3. For additional examples and more practice, see the Checkpoint 9 and Checkpoint 10A materials at the back of the student textbook.

Example 1

LuAnn has \$500 with which to open a savings account. She can open an account at Fredrico's Bank, which pays 7% interest, compounded monthly, or Money First Bank, which pays 7.25%, compounded quarterly. LuAnn plans to leave the money in the account, untouched, for ten years. In which account should she place the money? Justify your answer.

Solution: The obvious answer is that she should put the money in the account that will pay her the most interest over the ten years, but which bank is that? At both banks the principal (the initial value) is \$500. Fredrico's Bank pays 7% compounded monthly, which means the interest rate is $\frac{0.07}{12} \approx 0.00583$ each month. If LuAnn puts her money into Fredrico's Bank, after one month she will have:

$$500 + 500(0.00583) = 500(1.00583) \approx \$502.92.$$

To calculate the amount at the end of the second month, we must multiply by 1.00583 again, making the amount:

$$500(1.00583)^2 \approx \$505.85.$$

At the end of three months, the balance is:

$$500(1.00583)^3 \approx \$508.80.$$

This will happen every month for ten years, which is 120 months. At the end of 120 months, the balance will be:

$$500(1.00583)^{120} \approx \$1004.43.$$

Note that this last equation is an exponential function in the form $y = ab^x$, where y is the amount of money in the account and x is the number of months (in this case, 120 months). $a = 500$ is the starting value (at 0 months), and $b = 1.00583$ is the multiplier or growth rate for the account each month.

Example continues on next page →

Example continued from previous page.

A similar calculation is performed for Money First Bank. Its interest rate is higher, 7.25%, but it is only calculated and compounded quarterly. (Quarterly means four times each year, or every three months.) Hence, every quarter the bank calculates $\frac{0.0725}{4} = 0.018125$ interest. At the end of the first quarter, LuAnn would have:

$$500(1.018125) \approx \$509.06.$$

At the end of ten years (40 quarters) LuAnn would have:

$$500(1.018125)^{40} \approx \$1025.69.$$

Note that this last equation is an exponential function in the form $y = ab^x$, where y is the amount of money in the account and x is the number of quarters (in this case, 40 quarters). $a = 500$ is the starting value (at 0 quarters), and $b = 1.018125$ is the multiplier or growth rate for the account each quarter.

Since Money First Bank would pay her approximately \$21 more in interest than Fredrico's Bank, she should put her money in Money First Bank.

Example 2

Most homes appreciate in value, at varying rates, depending on the home's location, size, and other factors. But if a home is used as a rental, the Internal Revenue Service allows the owner to assume that it will depreciate in value. Suppose a house that costs \$150,000 is used as a rental property, and depreciates at a rate of 8% per year. What is the multiplier that will give the value of the house after one year? What is the value of the house after one year? What is the value after ten years? When will the house be worth half of its purchase price? Draw a graph of this situation.

Solution: Unlike interest, which increases the value of the house, depreciation takes value away. After one year, the value of the house is $150000 - 0.08(150000)$ which is the same as $150000(0.92)$. Therefore the multiplier is 0.92. After one year, the value of the house is $150000(0.92) = \$138{,}000$. After ten years, the value of the house will be $150000(0.92)^{10} \approx \$65{,}158.27$.

This last equation is an exponential function in the form $y = ab^x$, where y is the value of the house and x is the number of years. $a = 150000$ is the starting value (at 0 years), and $b = 0.92$ is the multiplier or growth factor (in this case, decay) each year.

To find when the house will be worth half of its purchase price, we need to determine when the value of the house reaches \$75,000. We just found that after ten years, the value is below \$75,000, so this situation occurs in less than ten years. To help answer this question, list the house's values in a table to see the depreciation.

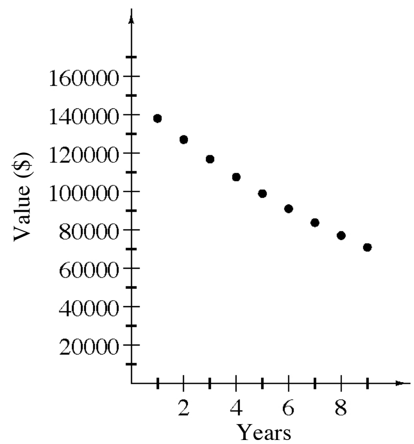
Example continues on next page →

Example continued from previous page.

From the table or graph, you can see that the house will be worth half its purchase price after 8 years.

Note: You can write the equation $75000 = 150000 \cdot 0.92^x$, but you will not have the mathematics to solve this equation for x until a future course. However, you can use the equation to find a more exact value: try different values for x in the equation, so the y-value gets closer and closer to \$75,000. At about 8.313 years the house's value is close to \$75,000.

# Years	House's value
1	138000
2	126960
3	116803.20
4	107458.94
5	98862.23
6	90953.25
7	83676.99
8	76982.83
9	70824.20



Example 3

Write an equation that represents the function in this table.

Week	Weight of Bacterial Culture (g)
1	756.00
2	793.80
3	833.49

The exponential function will have the form $y = ab^x$, where y is the weight of the bacterial culture, and x is the number of weeks. The multiplier, b, for the weight of the bacterial culture is 1.05 (because $793.80 \div 756 = 1.05$ and $833.49 \div 793.80 = 1.05$, etc.). The starting point, a is not given because we are not given the weight at Week 0. However, since the growth is 1.05 every week, we know that $(1.05) \cdot$ (weight at Week 0) = 756.00g. The weight at Week 0 is 720g, thus $a = 720$. We can now write the equation:

$$y = 720 \cdot 1.05^x,$$

where y is the weight of the bacterial culture (g), and x is the time (weeks).

Problems

1. In seven years, Seta's son Stu is leaving home for college. Seta hopes to save $8000 to pay for his first year. She has $5000 now and has found a bank that pays 7.75% interest, compounded daily. At this rate, will she have the money she needs for Stu's first year of college? If not, how much more does she need?

2. Eight years ago, Rudi thought that he was making a sound investment by buying $1000 worth of Pro Sports Management stock. Unfortunately, his investment depreciated steadily, losing 15% of its value each year. How much is the stock worth now? Justify your answer.

3. Based on each table below, find the equation of the exponential function $y = ab^x$.

a.

x	$f(x)$
0	1600
1	2000
2	2500
3	3125

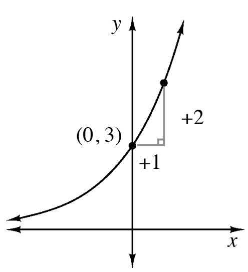
b.

x	y
1	40
2	32
3	25.6

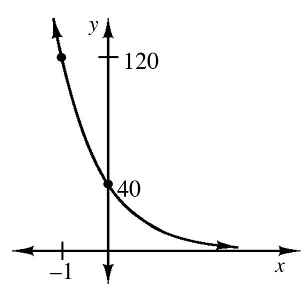
4. The new Bamo Super Ball has a rebound ratio of 0.97. If you dropped the ball from a height of 125 feet, how high will it bounce on the tenth bounce?

5. Based on each graph below, find the equation of the exponential function $y = ab^x$.

a.



b.



6. Fredrico's Bank will let you decide how often your interest will be computed, but with certain restrictions. If your interest is compounded yearly you can earn 8%. If your interest is compounded quarterly, you earn 7.875%. Monthly compounding earns a 7.75% interest rate, while weekly compounding earns a 7.625% interest rate. If your interest is compounded daily, you earn 7.5%. What is the best deal? Justify your answer.

7. Fully investigate the graph of the function $y = \left(\frac{3}{4}\right)^x + 4$. See Describing Functions (Lessons 1.1.3 through 1.2.2) in this *Parent Guide with Extra Practice* for information on how to fully describe the graph of a function.

Answers

1. Yes, she will have about \$8601.02 by then. The daily rate is $\frac{0.0775}{365} \approx 0.000212329$. Seven years is 2555 days, so we have $\$5000(1.000212329)^{2555} \approx \$8601.02$.

2. It is now only worth about \$272.49.

3. a. $y = 1600(1.25)^x$ b. $y = 50(0.8)^x$

4. About 92.18 feet.

5. a. $y = 3\left(\frac{5}{3}\right)^x$ b. $y = 40\left(\frac{1}{3}\right)^x$

6. The best way to do this problem is to choose any amount, and see how it grows over the course of one year. Taking \$100, after one year, 8% compounded yearly will yield \$108. 7.875% compounded quarterly yields \$108.11. 7.75% compounded monthly yields \$108.03. 7.625% compounded weekly yields \$107.91. 7.5% compounded daily yields \$107.79. Quarterly is the best.

7. This is a function that it is continuous and nonlinear (curved). It has a y-intercept of (0, 5), and no x-intercepts. The domain is all real values of x, and the range is all real values of $y > 4$. This function has a horizontal asymptote of $y = 4$, and no vertical asymptotes. It is an exponential function.

FRACTIONAL EXPONENTS 7.2.1

A fractional exponent is equivalent to an expression with roots or radicals.

For $x \neq 0$, $x^{a/b} = (x^a)^{1/b} = \sqrt[b]{x^a}$ or $x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a$.

Fractional exponents may also be used to solve equations containing exponents. For additional information, see the Math Notes box in Lesson 7.2.2.

Example 1

Rewrite each expression in radical form and simplify if possible.

a. $16^{5/4}$ b. $(-8)^{2/3}$

Solution:

a.

$$\begin{aligned} &16^{5/4} \\ &= (16^{1/4})^5 \\ &= (\sqrt[4]{16})^5 \\ &= (2)^5 \\ &= 32 \end{aligned}$$

OR

$$\begin{aligned} &16^{5/4} \\ &= (16^5)^{1/4} \\ &= (1{,}048{,}576)^{1/4} \\ &= \sqrt[4]{1{,}048{,}576} \\ &= 32 \end{aligned}$$

b.

$$\begin{aligned} &(-8)^{2/3} \\ &= \left((-8)^{1/3}\right)^2 \\ &= \left(\sqrt[3]{-8}\right)^2 \\ &= (-2)^2 \\ &= 4 \end{aligned}$$

OR

$$\begin{aligned} &(-8)^{2/3} \\ &= \left((-8)^2\right)^{1/3} \\ &= \left(64\right)^{1/3} \\ &= \sqrt[3]{64} \\ &= 4 \end{aligned}$$

Example 2

Simplify each expression. Each answer should contain no parentheses and no negative exponents.

a. $(144x^{-12})^{1/2}$ b. $\left(\frac{8x^7y^3}{x}\right)^{-1/3}$

Using the Power Property of Exponents, the Property of Negative Exponents, and the Property of Fractional Exponents:

a. $$(144x^{-12})^{1/2} = \left(\frac{144}{x^{12}}\right)^{1/2} = \sqrt{\frac{144}{x^{12}}} = \frac{12}{x^6}$$

b. $$\left(\frac{8x^7y^3}{x}\right)^{-1/3} = \left(\frac{x}{8x^7y^3}\right)^{1/3} = \left(\frac{1}{8x^6y^3}\right)^{1/3} = \sqrt[3]{\frac{1}{8x^6y^3}} = \frac{1}{2x^2y}$$

Example 3

Solve the following equations for x.

a. $x^7 = 42$ b. $3x^{12} = 132$

Solution: As with many equations, we need to isolate the variable (get the variable by itself), and then eliminate the exponent. This will require one of the Laws of Exponents, namely $(x^a)^b = x^{ab}$.

a.
$$\begin{aligned} x^7 &= 42 \\ (x^7)^{1/7} &= (42)^{1/7} \\ x^{7\cdot(1/7)} &= (42)^{1/7} \\ x^1 &= (42)^{1/7} \\ x &\approx 1.706 \end{aligned}$$

b.
$$\begin{aligned} 3x^{12} &= 132 \\ \frac{3x^{12}}{3} &= \frac{132}{3} \\ x^{12} &= 44 \\ (x^{12})^{1/12} &= (44)^{1/12} \\ x &= (44)^{1/12} \\ x &\approx \pm 1.371 \end{aligned}$$

The final calculation takes the seventh root of 42 in part (a) and the twelfth root of 44 in part (b). Notice that there is only one answer for part (a), where the exponent is odd, but there are two answers (±) in part (b) where the exponent is even. Even roots always produce two answers, a positive and a negative. Be sure that if the problem is a real-world application that both the positive and the negative results make sense before stating both as solutions. You may have to disregard one solution so that the answer is feasible.

Problems

Change each expression to radical form and simplify.

1. $(64)^{2/3}$
2. $16^{-1/2}$
3. $(-27)^{1/3}$

Simplify the following expressions as much as possible.

4. $(16a^8b^{12})^{3/4}$
5. $\frac{144^{1/2}x^{-3}}{(16^{3/4}x^7)^0}$
6. $\frac{a^{2/3}b^{-3/4}c^{7/8}}{a^{-1/3}b^{1/4}c^{1/8}}$

Solve the following equations for x.

7. $x^8 = 65{,}536$
8. $-5x^{-3} = \frac{25}{40}$
9. $3^{5x} = 9^{x-1}$
10. $\left(\frac{2}{3}\right)^x = 1$
11. $2(3x-5)^4 = 512$
12. $2^{4x-1} = 4$

Find the error in each of the following solutions. Then give the correct solution.

13.
$$\begin{aligned} 4(x+7)^6 &= 1392 \\ (x+7)^6 &= 348 \\ x+7 &= 58 \\ x &= 51 \end{aligned}$$

14.
$$\begin{aligned} 5^{4x+2} &= 10^{3x-1} \\ 5^{4x+2} &= 2\cdot 5^{3x-1} \\ 4x+2 &= 2(3x-1) \\ 4x+2 &= 6x-1 \\ 3 &= 2x \\ x &= 1.5 \end{aligned}$$

Answers

1. $(\sqrt[3]{64})^2 = 16$
2. $\frac{1}{\sqrt{16}} = \frac{1}{4}$
3. $\sqrt[3]{-27} = -3$
4. $8a^6b^9$
5. $\frac{12}{x^3}$
6. $\frac{ac^{3/4}}{b}$
7. $x = 4$
8. $x = -2$
9. $x = -\frac{2}{3}$
10. $x = 0$
11. $x = 3$
12. $x = \frac{3}{4}$
13. Both sides need to be raised to the $\frac{1}{6}$ (or the 6^{th} root taken), not divided by six. $x \approx -4.35$.
14. Since 5 and 10 cannot be written as the power of the same number, the only way to solve the equation now is by guess and check. $x \approx 11.75$. If you did not get this answer, do not worry about it now. The point of the problem is to spot the error. There is a second error: the 2 was not distributed in the fourth line.

Students write an equation of the form $y = ab^x$ that goes through two given points. (Equations of this form have an asymptote at $y = 0$.)

For additional information, see the Math Notes box in Lesson 9.3.1. For additional examples and more practice, see the Checkpoint 9 and Checkpoint 10A materials.

Example 1

Find a possible equation for an exponential function that passes through the points (0, 8) and $(4, \frac{1}{2})$.

Solution: Substitute the x- and y-coordinates of each pair of points into the general equation. Then solve the resulting system of two equations to determine a and b.

$y = ab^x$

Since $(x, y) = (0, 8)$

$8 = ab^0$

Since $b^0 = 1$,

$8 = a(1)$, or

$a = 8$.

$y = ab^x$

Since $(x, y) = (4, \frac{1}{2})$,

$\frac{1}{2} = ab^4$

Substituting $a = 8$ from the first equation into $\frac{1}{2} = ab^4$ from the second equation,

$$\frac{1}{2} = ab^4$$

But, $a = 8$,

$$\frac{1}{2} = 8b^4$$

$$\frac{1}{16} = b^4$$

$$\sqrt[4]{\frac{1}{16}} = \sqrt[4]{b^4}$$

$$\frac{1}{2} = b$$

Since a and b have been determined, we can now write the equation:

$$y = 8(\tfrac{1}{2})^x$$

Example 2

In the year 2000, Club Leopard was first introduced on the Internet. In 2004, it had 14,867 "leopards" (members). In 2007, the leopard population had risen to 22,610. Model this data with an exponential function and use the model to predict the leopard population in the year 2012.

Solution: We can call the year 2000 our time zero, or $x = 0$. Then 2004 is $x = 4$, and the year 2007 will be $x = 7$. This gives us two data points, (4, 14867) and (7, 22610).

To model with an exponential function we will use the equation $y = ab^x$ and substitute both coordinate pairs to obtain a system of two equations.

$$y = ab^x \qquad y = ab^x$$
$$14867 = ab^4 \qquad 22610 = ab^7$$

Preparing to use the Equal Values Method to solve the system of equations, we rewrite both equations in "$a =$" form:

$$14867 = ab^4 \qquad 22610 = ab^7$$
$$a = \frac{14867}{b^4} \qquad a = \frac{22610}{b^7}$$

Then by the Equal Values Method,

$$\frac{14867}{b^4} = \frac{22610}{b^7}$$
$$14867b^7 = 22610b^4$$
$$\frac{b^7}{b^4} = \frac{22610}{14867}$$
$$b^3 = \frac{22610}{14867}$$
$$\sqrt[3]{b^3} = \sqrt[3]{\tfrac{22610}{14867}}$$
$$b \approx 1.15$$

From the equations above,

$$a = \frac{14867}{b^4}$$

Since $b \approx 1.15$,

$$a = \frac{14867}{(1.15)^4}$$
$$a \approx 8500$$

Since $a \approx 8500$ and $b \approx 1.15$ we can write the equation: $y = 8500 \cdot 1.15^x$, where y represents the number of members, and x represents the number of years since 2000.

We will use the equation with $x = 12$ to predict the population in 2012.

$$y = 8500(1.15)^x$$
$$y = 8500(1.15)^{12}$$
$$y \approx 45477$$

Assuming the trend continues to the year 2012 as it has in the past, we predict the population in 2012 to be 45,477.

Problems

For each of the following pairs of points, find the equation of an exponential function with an asymptote $y = 0$ that passes through them.

1. $(0, 6)$ and $(3, 48)$
2. $(1, 21)$ and $(2, 147)$
3. $(-1, 72.73)$ and $(3, 106.48)$
4. $(-2, 351.5625)$ and $(3, 115.2)$
5. On a cold wintry day the temperature outside hovered at 0°C. Karen made herself a cup of cocoa, and took it outside where she would be chopping some wood. However, she decided to conduct a mini science experiment instead of drinking her cocoa, so she placed a thermometer in the cocoa and left it sitting next to her as she worked. She wrote down the time and the reading on the thermometer as shown in the table below.

Time since 1st reading	5	10	12	15
Temp (°C)	24.41°	8.51°	5.58°	2.97°

Find the equation of an exponential function of the form $y = ab^x$ that models this data.

Answers

1. $y = 6(2)^x$
2. $y = 3(7)^x$
3. $y = 80(1.1)^x$
4. $y = 225(0.8)^x$
5. Answers will vary, but should be close to $y = 70(0.81)^x$.

FACTORING QUADRATICS 8.1.1 through 8.1.4

Chapter 8 introduces students to rewriting quadratic expressions and solving quadratic equations. Quadratic functions are functions which can be rewritten in the form $y = ax^2 + bx + c$ (where $a \neq 0$) and when graphed, create a U-shaped curve called a parabola.

There are multiple methods that can be used to solve quadratic equations. One of them requires factoring the quadratic expression first. In Lessons 8.1.1 through 8.1.4, students factor quadratic expressions.

In previous chapters, students used algebra tiles to build "generic rectangles" of quadratic expressions. In the figure below, the length and width of the rectangle are $(x + 2)$ and $(x + 4)$. Since the area of a rectangle is given by (base)(height) = area, the area of the rectangle in the figure below can be expressed as a *product*, $(x + 2)(x + 4)$. But the small pieces of the rectangle also make up its area, so the area can be expressed as a *sum*, $4x + 8 + x^2 + 2x$, or $x^2 + 6x + 8$. Thus students wrote $(x + 2)(x + 4) = x^2 + 6x + 8$.

In the figure at right, the length and width of the rectangle, which are $(x + 2)$ and $(x + 4)$, are *factors* of the quadratic expression $x^2 + 6x + 8$, since $(x + 2)$ and $(x + 4)$ multiply together to produce the quadratic expression $x^2 + 6x + 8$. Notice that the $4x$ and the $2x$ are located diagonally from each other. They are like terms and can be combined and written as $6x$.

$+\ 4$	$4x$	8
x	x^2	$2x$
	x	$+\ 2$

The factors of $x^2 + 6x + 8$ are $(x + 2)$ and $(x + 4)$.

The ax^2 term and the c term are always diagonal to one another in a generic rectangle. In this example, the ax^2 term is $(1x^2)$ and the c term is the constant 8; the product of this diagonal is $1x^2 \cdot 8 = 8x^2$. The two x-terms make up the other diagonal and can be combined into a sum since they are like terms. The b of a quadratic expression is the *sum* of the coefficients of these factors: $2x + 4x = 6x$, so $b = 6$. The product of this other diagonal is $(2x)(4x) = 8x^2$. *Note that the products of the two diagonals are always equivalent.* In the textbook, students may nickname this rule "Casey's Rule," after the fictional character Casey in problem 8-4.

To factor a quadratic expression, students need to identify the coefficients of the two x-terms so that the products of the two diagonals are equivalent, and also the sum of the two x-terms is b. Students can use a "diamond problem" to help organize their sums and products. For more information on using a diamond problem and generic rectangle to factor quadratic expressions, see the Math Notes box in Lesson 8.1.4.

For additional information, see the Math Notes boxes in Lessons 8.1.1 through 8.1.4. For additional examples and more practice, see the Checkpoint 10B materials at the back of the student textbook.

Example 1

Factor $x^2 + 7x + 12$.

Sketch a generic rectangle.

Place the x^2 and the 12 along one diagonal.

	12
x^2	

Find two terms whose product is $12x^2$ and whose sum is $7x$. In this case, $3x$ and $4x$. (Students are familiar with this situation as a "diamond problem" from Chapter 1.)

$\mathbf{3x}$	12
x^2	$\mathbf{4x}$

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

+ 3	$3x$	12
x	x^2	$4x$
	x +	4

Write the sum as a product (factored form).

$x^2 + 7x + 12 = (x + 3)(x + 4)$

Example 2

Factor $x^2 + 7x - 30$.

Sketch a generic rectangle.
Place the x^2 and the –30 along one diagonal.

	–30
x^2	

Find two terms whose product is $-30x^2$ and whose sum is $7x$. In this case, $-3x$ and $10x$.

$\mathbf{-3x}$	–30
x^2	$\mathbf{10x}$

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

–3	$-3x$	–30
x	x^2	$10x$
	x +	10

Write the sum as a product (factored form).

$x^2 + 7x - 30 = (x - 3)(x + 10)$

Example 3

Factor $x^2 - 15x + 56$.

Sketch a generic rectangle.
Place the x^2 and the 56 along one diagonal.

	56
x^2	

Find two terms whose product is $56x^2$ and whose sum is $-15x$. Write these terms as the other diagonal.

$-8x$	56
x^2	**$-7x$**

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

-8	$-8x$	56
x	x^2	$-7x$
	x	-7

Write the sum as a product (factored form).

$$x^2 - 15x + 56 = (x - 7)(x - 8)$$

Example 4

Factor $12x^2 - 19x + 5$.

Sketch a generic rectangle.

Place the $12x^2$ and the 5 along one diagonal.

$-15x$	5
$12x^2$	**$-4x$**

→

-5	$-15x$	5
$4x$	$12x^2$	$-4x$
	$3x$	-1

Find two terms whose product is $60x^2$ and whose sum is $-19x$. Write these terms as the other diagonal.

Find the base and height of the rectangle. Check the signs of the factors.

Write the sum as a product (factored form). $(3x - 1)(4x - 5) = 12x^2 - 19x + 5$

Example 5

Factor $3x^2 + 21x + 36$.

Note: If a common factor appears in all the terms, it should be factored out first.
For example, $3x^2 + 21x + 36 = 3(x^2 + 7x + 12)$.

Then $x^2 + 7x + 12$ can be factored in the usual way, as in Example.
$x^2 + 7x + 12 = (x + 3)(x + 4)$.

Then, since the expression $3x^2 + 21x + 36$ has a factor of 3,
$3x^2 + 21x + 36 = 3(x^2 + 7x + 12) = 3(x + 3)(x + 4)$.

Problems

1. $x^2 + 5x + 6$	2. $2x^2 + 5x + 3$	3. $3x^2 + 4x + 1$	4. $3x^2 + 30x + 75$
5. $x^2 + 15x + 44$	6. $x^2 + 7x + 6$	7. $2x^2 + 22x + 48$	8. $x^2 + 4x - 32$
9. $4x^2 + 12x + 9$	10. $24x^2 + 22x - 10$	11. $x^2 + x - 72$	12. $3x^2 - 20x - 7$
13. $x^3 - 11x^2 + 28x$	14. $2x^2 + 11x - 6$	15. $2x^2 + 5x - 3$	16. $x^2 - 3x - 10$
17. $4x^2 - 12x + 9$	18. $3x^2 + 2x - 5$	19. $6x^2 - x - 2$	20. $9x^2 - 18x + 8$

Answers

1. $(x + 2)(x + 3)$	2. $(x + 1)(2x + 3)$	3. $(3x + 1)(x + 1)$	4. $3(x + 5)(x + 5)$
5. $(x + 11)(x + 4)$	6. $(x + 6)(x + 1)$	7. $2(x + 8)(x + 3)$	8. $(x + 8)(x - 4)$
9. $(2x + 3)(2x + 3)$	10. $2(3x - 1)(4x + 5)$	11. $(x - 8)(x + 9)$	12. $(x - 7)(3x + 1)$
13. $x(x - 4)(x - 7)$	14. $(x + 6)(2x - 1)$	15. $(x + 3)(2x - 1)$	16. $(x - 5)(x + 2)$
17. $(2x - 3)(2x - 3)$	18. $(3x + 5)(x - 1)$	19. $(2x + 1)(3x - 2)$	20. $(3x - 4)(3x - 2)$

FACTORING SHORTCUTS 8.1.5

Although most factoring problems can be done with generic rectangles, there are two special factoring patterns that, if recognized, can be done by sight. The two patterns are known as the **Difference of Squares** and **Perfect Square Trinomials**. The general patterns are as follows:

Difference of Squares: $a^2x^2 - b^2y^2 = (ax + by)(ax - by)$
Perfect Square Trinomial: $a^2x^2 + 2abxy + b^2y^2 = (ax + by)^2$

Example 1

Difference of Squares	**Perfect Square Trinomials**
$x^2 - 49 = (x+7)(x-7)$	$x^2 - 10x + 25 = (x-5)^2$
$4x^2 - 25 = (2x-5)(2x+5)$	$9x^2 + 12x + 4 = (3x+2)^2$
$x^2 - 36 = (x+6)(x-6)$	$x^2 - 6x + 9 = (x-3)^2$
$9x^2 - 1 = (3x-1)(3x+1)$	$4x^2 + 20x + 25 = (2x+5)^2$

Example 2

Sometimes removing a common factor reveals one of the special patterns:

$8x^2 - 50y^2 \Rightarrow 2(4x^2 - 25y^2) \Rightarrow 2(2x + 5y)(2x - 5y)$

$12x^2 + 12x + 3 \Rightarrow 3(4x^2 + 4x + 1) \Rightarrow 3(2x + 1)^2$

Problems

Factor each difference of squares.

1. $x^2 - 16$	2. $x^2 - 25$	3. $64m^2 - 25$
4. $4p^2 - 9q^2$	5. $9x^2y^2 - 49$	6. $x^4 - 25$
7. $64 - y^2$	8. $144 - 25p^2$	9. $9x^4 - 4y^2$

Factor each perfect square trinomial.

10. $x^2 + 4x + 4$	11. $y^2 + 8y + 16$	12. $m^2 - 10m + 25$
13. $x^2 - 8x + 16$	14. $a^2 + 8ab + 16b^2$	15. $36x^2 + 12x + 1$
16. $25x^2 - 30xy + 9y^2$	17. $9x^2y^2 - 6xy + 1$	18. $49x^2 + 1 + 14x$

Factor completely.

19. $9x^2 - 16$	20. $9x^2 + 24x + 16$	21. $9x^2 - 36$
22. $2x^2 + 8xy + 8y^2$	23. $x^2y + 10xy + 25y$	24. $8x^2 - 72$
25. $4x^3 - 9x$	26. $4x^2 - 8x + 4$	27. $2x^2 + 8$

Answers

1. $(x + 4)(x - 4)$	2. $(x + 5)(x - 5)$	3. $(8m + 5)(8m - 5)$
4. $(2p + 3q)(2p - 3q)$	5. $(3xy + 7)(3xy - 7)$	6. $(x^2 + 5)(x^2 - 5)$
7. $(8 + y)(8 - y)$	8. $(12 + 5p)(12 - 5p)$	9. $(3x^2 + 2y)(3x^2 - 2y)$
10. $(x + 2)^2$	11. $(y + 4)^2$	12. $(m - 5)^2$
13. $(x - 4)^2$	14. $(a + 4b)^2$	15. $(6x + 1)^2$
16. $(5x - 3y)^2$	17. $(3xy - 1)^2$	18. $(7x + 1)^2$
19. $(3x + 4)(3x - 4)$	20. $(3x + 4)^2$	21. $9(x + 2)(x - 2)$
22. $2(x + 2y)^2$	23. $y(x + 5)^2$	24. $8(x + 3)(x - 3)$
25. $x(2x + 3)(2x - 3)$	26. $4(x - 1)^2$	27. $2(x^2 + 4)$

USING THE ZERO PRODUCT PROPERTY 8.2.2 and 8.2.3

The graph of a quadratic function, a parabola, is a symmetrical curve. Its highest or lowest point is called the vertex. The graph is created by using the equation $y = ax^2 + bx + c$. Students have been graphing parabolas by substituting values for x and solving for y. This can be a tedious process, especially if an appropriate range of x-values is not known. If only a quick sketch of the parabola is needed, one possible method is to find the x-intercepts first, then find the vertex and/or the y-intercept. To find the x-intercepts, substitute 0 for y and solve the quadratic equation, $0 = ax^2 + bx + c$. Students will learn multiple methods to solve quadratic equations in this chapter and in Chapter 9. One method to solve quadratic equations uses the Zero Product Property, that is, solving by factoring. This method uses two ideas:

(1) When the product of two or more numbers is zero, then one of the numbers must be zero.

(2) Some quadratic expressions can be factored into the product of two binomials.

For additional information see the Math Notes box in Lesson 8.2.2.

Example 1

Find the x-intercepts of $y = x^2 + 6x + 8$.

The x-intercepts are located on the graph where $y = 0$, so write the quadratic expression equal to zero, then solve for x. $\quad x^2 + 6x + 8 = 0$

Factor the quadratic expression. $\quad (x + 4)(x + 2) = 0$

Set each factor equal to 0. $\quad (x + 4) = 0$ or $(x + 2) = 0$

Solve each equation for x. $\quad x = -4$ or $x = -2$

The x-intercepts are $(-4, 0)$ and $(-2, 0)$.

You can check your answers by substituting them into the original equation.

$(-4)^2 + 6(-4) + 8 \Rightarrow 16 - 24 + 8 \Rightarrow 0$
$(-2)^2 + 6(-2) + 8 \Rightarrow 4 - 12 + 8 \Rightarrow 0$

Example 2

Solve $2x^2 + 7x - 15 = 0$.

Factor the quadratic expression. $(2x - 3)(x + 5) = 0$

Set each factor equal to 0. $(2x - 3) = 0$ or $(x + 5) = 0$

Solve for each x. $2x = 3$ or $x = -5$

$x = \frac{3}{2}$ or $x = -5$

Example 3

If the quadratic equation does not equal 0, rewrite it algebraically so that it does, then use the Zero Product Property.

Solve $2 = 6x^2 - x$.

Set the equation equal to 0.

$$2 = 6x^2 - x$$
$$0 = 6x^2 - x - 2$$

Factor the quadratic expression.

$$0 = (2x + 1)(3x - 2)$$

Solve each equation for x.

$$(2x + 1) = 0 \quad \text{or} \quad (3x - 2) = 0$$
$$2x = -1 \quad \text{or} \quad 3x = 2$$
$$x = -\frac{1}{2} \qquad x = \frac{2}{3}$$

Example 4

Solve $9x^2 - 6x + 1 = 0$.

Factor the quadratic expression.

$$9x^2 - 6x + 1 = 0$$
$$(3x - 1)(3x - 1) = 0$$

Solve each equation for x. Notice the factors are the same so there will be only one solution.

$$(3x - 1) = 0$$
$$3x = 1$$
$$x = \frac{1}{3}$$

Problems

Solve for x.

1. $x^2 - x - 12 = 0$	2. $3x^2 - 7x - 6 = 0$	3. $x^2 + x - 20 = 0$
4. $3x^2 + 11x + 10 = 0$	5. $x^2 + 5x = -4$	6. $6x - 9 = x^2$
7. $6x^2 + 5x - 4 = 0$	8. $x^2 - 6x + 8 = 0$	9. $6x^2 - x - 15 = 0$
10. $4x^2 + 12x + 9 = 0$	11. $x^2 - 12x = 28$	12. $2x^2 + 8x + 6 = 0$
13. $2 + 9x = 5x^2$	14. $2x^2 - 5x = 3$	15. $x^2 = 45 - 4x$

Answers

1. $x = 4$ or -3	2. $x = -\frac{2}{3}$ or 3	3. $x = -5$ or 4
4. $x = -\frac{5}{3}$ or -2	5. $x = -4$ or -1	6. $x = 3$
7. $x = -\frac{4}{3}$ or $\frac{1}{2}$	8. $x = 4$ or 2	9. $x = -\frac{3}{2}$ or $\frac{5}{3}$
10. $x = -\frac{3}{2}$	11. $x = 14$ or -2	12. $x = -1$ or -3
13. $x = -\frac{1}{5}$ or 2	14. $x = -\frac{1}{2}$ or 3	15. $x = 5$ or -9

In Lesson 8.2.3, students found that if the equation of a parabola is written in **graphing form**: $f(x) = (x - h)^2 + k$ then the vertex can easily be seen as (h, k). For example, for the parabola $f(x) = (x + 3)^2 - 1$ the vertex is $(-3, -1)$. Students can then set the function equal to zero to find the x-intercepts: solve $0 = (x + 3)^2 - 1$ to find the x-intercepts. For help in solving this type of equation, see the Lesson 8.2.3 Resource Page, available at cpm.org. Students can set $x = 0$ to find the y-intercepts: $y = (0 + 3)^2 - 1$.

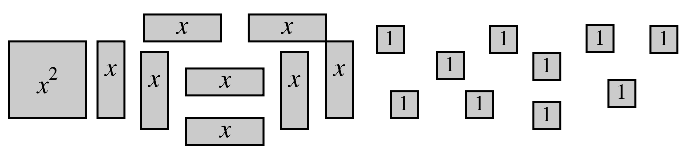
When the equation of the parabola is given in standard form: $f(x) = x^2 + bx + c$, then using the process of **completing the square** can be used to convert standard form into graphing form. Algebra tiles are used to help visualize the process.

For additional examples and practice with graphing quadratic functions, see the Checkpoint 11 materials at the back of the student textbook.

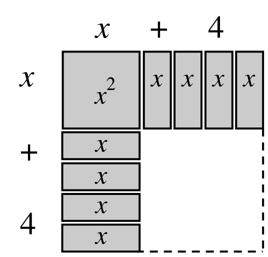
Example 1 (Using algebra tiles)

Complete the square to change $f(x) = x^2 + 8x + 10$ into graphing form, identify the vertex and y-intercept, and draw a graph.

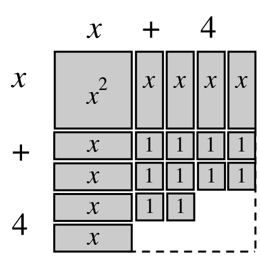
$f(x) = x^2 + 8x + 10$ would look like this:



Arrange the tiles as shown in the picture at right to make a square.



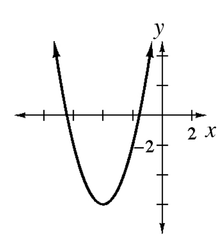
16 small unit tiles are needed to fill in the corner, but only ten unit tiles are available. Show the 10 small square tiles and draw the outline of the whole square.



The **complete square** would have length and width both equal to $(x + 4)$, so the complete square can be represented by the quadratic expression $(x + 4)^2$. But the tiles from $x^2 + 8x + 10$ do not form a complete square—the expression $x^2 + 8x + 10$ has six fewer tiles than a complete square. So $x^2 + 8x + 10$ is a complete square minus 6, or $(x + 4)^2$ minus 6. That is,

$$x^2 + 8x + 10 = (x + 4)^2 - 6.$$

From graphing form, the vertex is at $(h, k) = (-4, -6)$. The y-intercept is where $x = 0$, so $y = (0 + 4)^2 - 6 = 10$. The y-intercept is $(0, 10)$, and the graph is shown at right.



Example 2 (Using the general process)

Complete the square to change $f(x) = x^2 + 5x + 2$ into graphing form. Identify the vertex and y-intercept, and draw a graph.

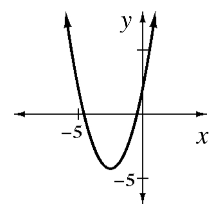
Rewrite the expression as: $f(x) = x^2 + 5x + 2$
$$f(x) = (x^2 + 5x + ?) + 2$$

Make $(x^2 + 5x + ?)$ into a perfect square by taking half of the x-term coefficient and squaring it: $(\frac{5}{2})^2 = \frac{25}{4}$. Then $(x^2 + 5x + \frac{25}{4})$ is a perfect square trinomial.

We need to write an equivalent function with $+\frac{25}{4}$, but if we add $\frac{25}{4}$, we also need to subtract it:

$f(x) = x^2 + 5x + 2$ which can be rewritten as: $f(x) = (x + \frac{5}{2})^2 - \frac{17}{4}$.

$f(x) = (x^2 + 5x + \frac{25}{4}) + 2 - \frac{25}{4}$



The function is now in graphing form.
The vertex is $(-\frac{5}{2}, -\frac{17}{4})$ or $(-2.5, -4.25)$.
The y-intercept is where $x = 0$.
Thus, $y = (0 + 2.5)^2 - 4.25 = 2$ and the y-intercept is $(0, 2)$.

Alternatively, if the process above is unclear, draw a generic rectangle of $x^2 + 5x + 2$ and imagine algebra tiles.

$2.5x$	
x^2	$2.5x$

There should be $(-2.5)^2 = 6.25$ tiles in the upper right corner to complete the square. But the expression $x^2 + 5x + 2$ only provides 2 unit tiles. So there are 4.25 unit tiles missing. Thus, 4.25 is missing from the rectangle below:

$+2.5$	$2.5x$	
x	x^2	$2.5x$
	x	$+2.5$

That is, $(x + 2.5)^2$ minus 4.25 unit tiles is the equivalent of $x^2 + 5x + 2$, or,
$x^2 + 5x + 2 = (x + 2.5)^2 - 4.25$.

Problems

Complete the square to write each equation in graphing form. Then state the vertex.

1. $f(x) = x^2 + 6x + 7$
2. $f(x) = x^2 + 4x + 11$
3. $f(x) = x^2 + 10x$
4. $f(x) = x^2 + 7x + 2$
5. $f(x) = x^2 - 6x + 9$
6. $f(x) = x^2 + 3$
7. $f(x) = x^2 - 4x$
8. $f(x) = x^2 + 2x - 3$
9. $f(x) = x^2 + 5x + 1$
10. $f(x) = x^2 - \frac{1}{3}x$

Answers

1. $f(x) = (x + 3)^2 - 2$; $(-3, -2)$
2. $f(x) = (x + 2)^2 + 7$; $(-2, 7)$
3. $f(x) = (x + 5)^2 - 25$; $(-5, -25)$
4. $f(x) = (x + 3.5)^2 - 10.25$; $(-3.5, -10.25)$
5. $f(x) = (x - 3)^2$; $(3, 0)$
6. $f(x) = x^2 + 3$; $(0, 3)$
7. $f(x) = (x - 2)^2 - 4$; $(2, -4)$
8. $f(x) = (x + 1)^2 - 4$; $(-1, -4)$
9. $f(x) = (x + \frac{5}{2})^2 - \frac{21}{4}$; $(-\frac{5}{2}, -\frac{21}{4})$
10. $f(x) = (x - \frac{1}{6})^2 - \frac{1}{36}$; $(\frac{1}{6}, -\frac{1}{36})$

USING THE QUADRATIC FORMULA 9.1.2 and 9.1.3

When a quadratic equation is not factorable, another method is needed to solve for x. The Quadratic Formula can be used to calculate the roots of a quadratic function, that is, the x-intercepts of the parabola. The Quadratic Formula can be used with any quadratic equation, factorable or not. There may be two, one, or no solutions, depending on whether the parabola intersects the x-axis twice, once, or not at all.
The solution(s) to any quadratic equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The $\pm$ symbol is read as "plus or minus." It is shorthand notation that tells you to calculate the formula twice, once with + and again with – to get both x-values.

To use the formula, the quadratic equation must be written in *standard form*: $ax^2 + bx + c = 0$. This is necessary to correctly identify the values of a, b, and c. Once the equation is in standard form and equal to 0, a is the coefficient of the x^2-term, b is the coefficient of the x-term and c is the constant term.

For additional information, see the Math Notes boxes in Lessons 9.1.1 through 9.1.4 and 10.2.4.

Example 1

Solve $2x^2 - 5x - 3 = 0$.

Identify a, b, and c. Watch your signs carefully.	$a = 2$, $b = -5$, $c = -3$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute a, b, and c into the formula and do the initial calculations.	$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$
	$x = \frac{5 \pm \sqrt{25 - (-24)}}{4}$
Simplify the $\sqrt{\ }$.	$x = \frac{5 \pm \sqrt{49}}{4}$
Calculate both values of x.	$x = \frac{5+7}{4} = \frac{12}{4} = 3$ or $x = \frac{5-7}{4} = \frac{-2}{4} = -\frac{1}{2}$

The solutions are $x = 3$ or $x = -\frac{1}{2}$.

Example 2

Solve $3x^2 + 5x + 1 = 0$.

Identify a, b, and c. $a = 3$, $b = 5$, $c = 1$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{6}$$

Simplify the $\sqrt{}$.

$$x = \frac{-5 \pm \sqrt{13}}{6}$$

The solutions are $x = \frac{-5 + \sqrt{13}}{6} \approx -0.23$ or $x = \frac{-5 - \sqrt{13}}{6} \approx -1.43$.

Example 3

Solve $25x^2 - 20x + 4 = 0$.

Identify a, b, and c. $a = 25$, $b = -20$, $c = 4$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(25)(4)}}{2(25)}$$

$$x = \frac{20 \pm \sqrt{400 - 400}}{50}$$

Simplify the $\sqrt{}$.

$$x = \frac{20 \pm \sqrt{0}}{50}$$

This quadratic equation has only one solution: $x = \frac{2}{5}$.

Example 4

Solve $x^2 + 4x + 10 = 0$.

Identify a, b, and c. $a = 1$, $b = 4$, $c = 10$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations. $x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(10)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{16 - 40}}{2}$

Simplify the $\sqrt{}$. $x = \frac{-4 \pm \sqrt{-24}}{2}$

It is impossible to take the square root of a negative number; therefore this quadratic equation has no real solutions.

Example 5

Solve $(3x + 1)(x + 2) = 1$.

Rewrite the equation in standard form. That is, rewrite the product as a sum and then set the equation equal to zero.

$$(3x + 1)(x + 2) = 1$$
$$3x^2 + 7x + 2 = 1$$
$$3x^2 + 7x + 1 = 0$$

Identify a, b, and c. $a = 3$, $b = 7$, $c = 1$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations. $x = \frac{-(7) \pm \sqrt{(7)^2 - 4(3)(1)}}{2(3)}$

$x = \frac{-7 \pm \sqrt{49 - 12}}{6}$

Simplify. $x = \frac{-7 \pm \sqrt{37}}{6}$

The solutions are $x = \frac{-7 \pm \sqrt{37}}{6}$, or, $x \approx -0.15$ or $x \approx -2.18$.

Example 6

Solve $3x^2+6x+1=0$.

Identify *a, b*, and *c*. $a=3, b=6, c=1$

Write the Quadratic Formula. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Substitute *a, b*, and *c* into the formula and do the initial calculations. $x=\frac{-(6)\pm\sqrt{(6)^2-4(3)(1)}}{2(3)}$

$x=\frac{-6\pm\sqrt{36-12}}{6}$

Simplify. $x=\frac{-6\pm\sqrt{24}}{6}$

The solutions are $x=\frac{-6\pm\sqrt{24}}{6}$, or, $x\approx-1.82$ or $x\approx-0.18$.

The Math Notes box in Lesson 9.1.4 describes another form of the expression $\frac{-6\pm\sqrt{24}}{6}$ that can be written by simplifying the square root. The result is equivalent to the exact values above.

Factor the $\sqrt{24}$, then simplify by taking the square root of 4. $\sqrt{24}=\sqrt{4}\sqrt{6}=2\sqrt{6}$

Simplify the fraction by dividing every term by 2. $x=\frac{-6\pm2\sqrt{6}}{6}$

$x=\frac{-3\pm\sqrt{6}}{3}$

Problems

Solve each equation by using the Quadratic Formula.

1. $x^2 - x - 2 = 0$	2. $x^2 - x - 3 = 0$	3. $-3x^2 + 2x + 1 = 0$
4. $-2 - 2x^2 = 4x$	5. $7x = 10 - 2x^2$	6. $-6x^2 - x + 6 = 0$
7. $6 - 4x + 3x^2 = 8$	8. $4x^2 + x - 1 = 0$	9. $x^2 - 5x + 3 = 0$
10. $0 = 10x^2 - 2x + 3$	11. $x(-3x + 5) = 7x - 10$	12. $(5x + 5)(x - 5) = 7x$

Answers

1. $x = 2$ or -1	2. $x = \frac{1 \pm \sqrt{13}}{2}$ ≈ 2.30 or -1.30	3. $x = -\frac{1}{3}$ or 1
4. $x = -1$	5. $x = \frac{-7 \pm \sqrt{129}}{4}$ ≈ 1.09 or -4.59	6. $x = \frac{1 \pm \sqrt{145}}{-12}$ ≈ -1.09 or 0.92
7. $x = \frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3}$ ≈ 1.72 or -0.39	8. $x = \frac{-1 \pm \sqrt{17}}{8}$ ≈ 0.39 or -0.64	9. $x = \frac{5 \pm \sqrt{13}}{2}$ ≈ 4.30 or 0.70
10. no solution	11. $x = \frac{2 \pm \sqrt{124}}{-6} = \frac{1 \pm \sqrt{31}}{-3}$ ≈ -2.19 or 1.52	12. $x = \frac{27 \pm \sqrt{1229}}{10}$ ≈ 6.21 or -0.81

SOLVING ONE-VARIABLE INEQUALITIES 9.2.1 and 9.2.2

To **solve an inequality in one variable,** first change it to an equation and solve. Place the solution, called a "boundary point," on a number line. This point separates the number line into two regions. The boundary point is included in the solution for situations that involve ≥ or ≤, and excluded from situations that involve strictly > or <. On the number line graph, boundary points that are included in the solutions are shown with a solid filled-in circle, and excluded solutions are shown with an open circle. Next, choose a number from within each region separated by the boundary point, and check if the number is true or false in the original inequality. If it is true, then every number in that region is a solution to the inequality. If it is false, then no number in that region is a solution to the inequality.

For additional information, see the Math Notes boxes in Lessons 9.2.1 and 9.3.2.

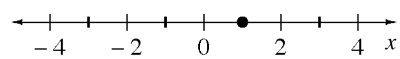
Example 1

Solve: $3x - (x + 2) \geq 0$

Change to an equation and solve.

$$3x - (x+2) = 0$$
$$3x - x - 2 = 0$$
$$2x = 2$$
$$x = 1$$

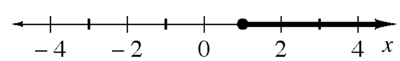
Place the solution (boundary point) on the number line. Because $x = 1$ is also a solution to the inequality (≥), we use a solid dot.



Test a number from each side of the boundary point in the original inequality.

Test $x = 0$	Test $x = 3$
$3 \cdot 0 - (0+2) \geq 0$	$3 \cdot 3 - (3+2) \geq 0$
$-2 \geq 0$	$4 \geq 0$
False	True

The solution is $x \geq 1$.



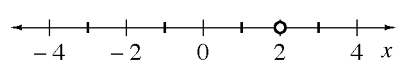
Example 2

Solve: $-x + 6 > x + 2$

Change to an equation and solve.

$$-x + 6 = x + 2$$
$$-2x = -4$$
$$x = 2$$

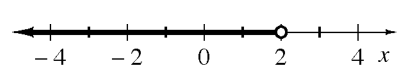
Place the solution (boundary point) on the number line. Because the original problem is a strict inequality (>), $x = 2$ is not a solution, so we use an open dot.



Test a number from each side of the boundary point in the original inequality.

Test $x = 0$	Test $x = 4$
$-0 + 6 > 0 + 2$	$-4 + 6 > 4 + 2$
$6 > 2$	$2 > 6$
True	False

The solution is $x < 2$.



Problems

Solve each inequality.

1.	$4x-1 \geq 7$	2.	$2(x-5) \leq 8$	3.	$3-2x < x+6$
4.	$\frac{1}{2}x > 5$	5.	$3(x+4) > 12$	6.	$2x-7 \leq 5-4x$
7.	$3x+2 < 11$	8.	$4(x-6) \geq 20$	9.	$\frac{1}{4}x < 2$
10.	$12-3x > 2x+1$	11.	$\frac{x-5}{7} \leq -3$	12.	$3(5-x) \geq 7x-1$
13.	$3y-(2y+2) \leq 7$	14.	$\frac{m+2}{5} < \frac{2m}{3}$	15.	$\frac{m-2}{3} \geq \frac{2m+1}{7}$

Answers

1.	$x \geq 2$	2.	$x \leq 9$	3.	$x > -1$
4.	$x > 10$	5.	$x > 0$	6.	$x \leq 2$
7.	$x < 3$	8.	$x \geq 11$	9.	$x < 8$
10.	$x < \frac{11}{5}$	11.	$x \leq -16$	12.	$x \leq 1.6$
13.	$y \leq 9$	14.	$m > \frac{6}{7}$	15.	$m \geq 17$

GRAPHING TWO-VARIABLE INEQUALITIES 9.3.1 and 9.3.2

To graph the solutions to an inequality in two variables, first graph the corresponding equation. This graph is the boundary line (or curve), since all points that make the inequality true lie on one side or the other of the line. Before you graph the equation, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed. If the inequality symbol is either ≤ or ≥, then the boundary line is part of the inequality and it must be solid. If the inequality symbol is either < or >, then the boundary line is dashed.

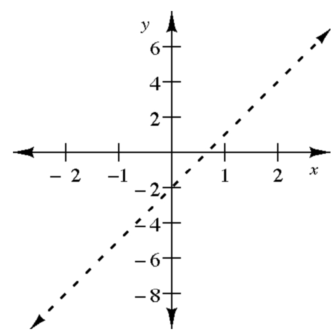
Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all values that make the inequality true. Choose a point not on the boundary line. Substitute this point into the *original* inequality. If the inequality is true for the test point, then shade the graph on this side of the boundary line. If the inequality is false for the test point, then shade the opposite side of the line.

The shaded portion represents all the solutions to the original inequality.

Caution: If you need to rearrange the inequality in order to graph it, such as putting it in slope-intercept form, always use the *original* inequality to test a point, not the rearranged form.

For additional information, see the Math Notes box in Lesson 9.4.1.

Example 1

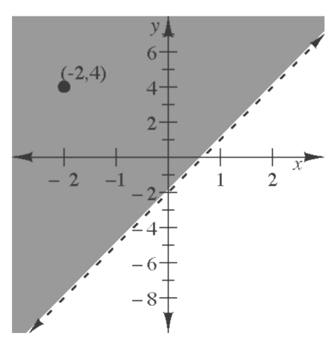


Graph the solutions to the inequality $y > 3x - 2$.

First, graph the line $y = 3x - 2$, but draw it dashed since > means the boundary line is not part of the solution. For example, the point $(0, -2)$ is on the boundary line, but it is not a solution to the inequality because $-2 \not> 3(0)-2$ or $-2 \not> -2$.

Next, test a point that is not on the boundary line. For this example, use the point $(-2, 4)$.

$4 > 3(-2) - 2$, so $4 > -8$ which is a true statement.



Since the inequality is true for this test point, shade the region containing the point $(-2, 4)$. All of the coordinate pairs that are solutions lie in the shaded region.

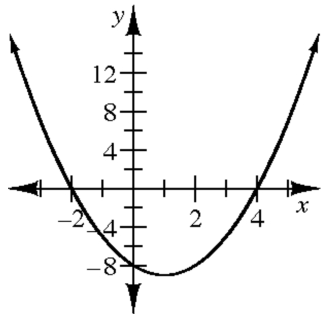
Example 2

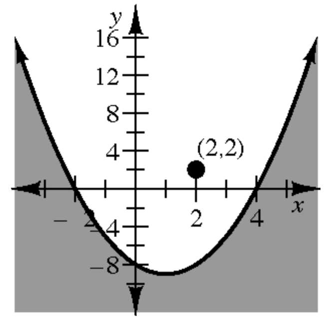
Graph the solutions to the inequality $y \leq x^2 - 2x - 8$.

First, graph the parabola $y = x^2 - 2x - 8$ and draw it solid, since $\leq$ means the boundary curve is part of the solution.

Next, test the point (2, 2) above the boundary curve.

$2 \leq 2^2 - 2 \cdot 2 - 8$, so $2 \leq -8$





Since the inequality is false for this test point above the curve, shade below the boundary curve. The solutions are the shaded region.

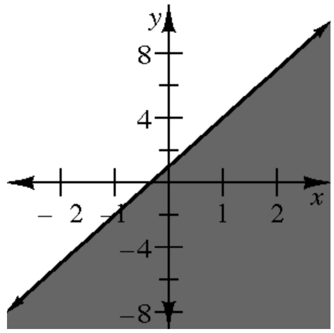
Problems

Graph the solutions to each of the following inequalities on a separate set of axes.

1. $y \leq 3x + 1$	2. $y \geq -2x + 3$	3. $y > 4x - 2$
4. $y < -3x - 5$	5. $y \leq 3$	6. $x > 1$
7. $y > \frac{2}{3}x + 8$	8. $y < -\frac{3}{5}x - 7$	9. $3x + 2y \geq 7$
10. $-4x + 2y < 3$	11. $y \geq x^2 - 3$	12. $y \leq x^2 + 2x$
13. $y < 4 - x^2$	14. $y \leq \|x + 2\|$	15. $y \geq -\|x\| + 3$

Answers

1.



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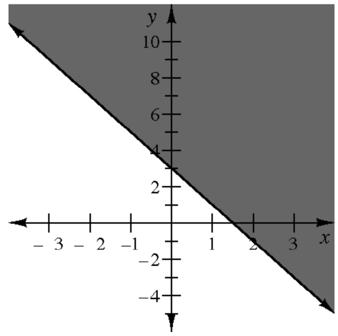

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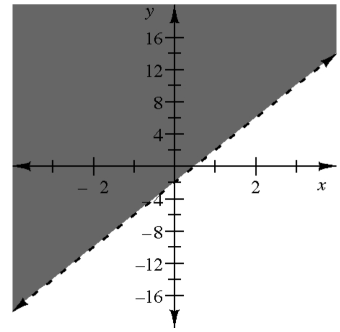

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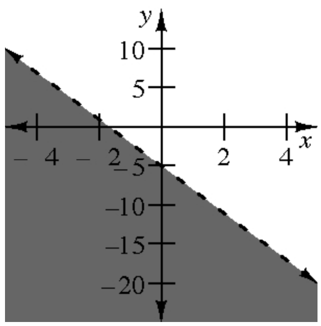

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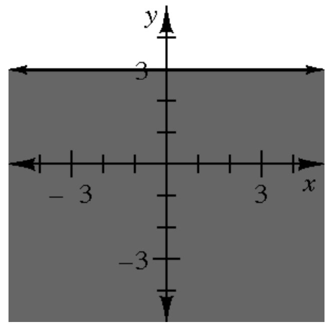

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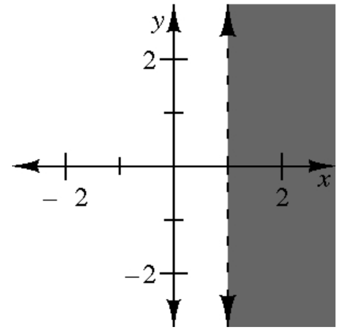

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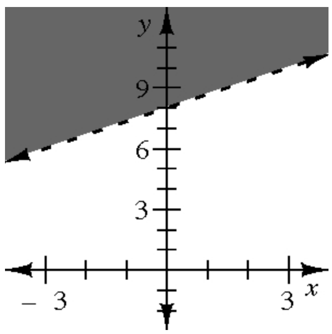

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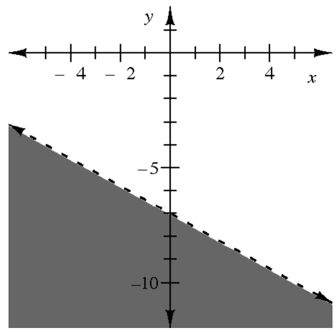

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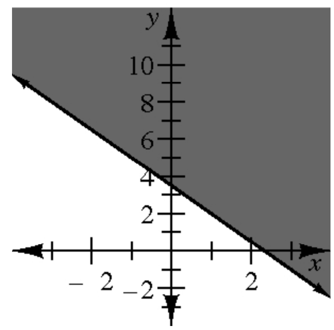

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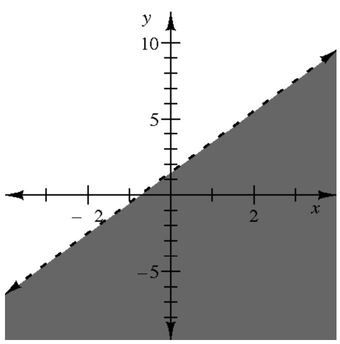

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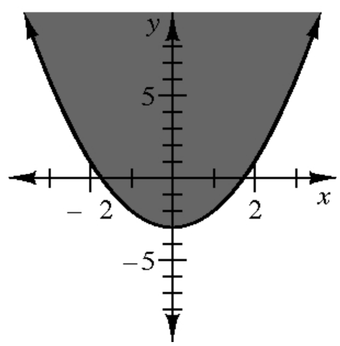

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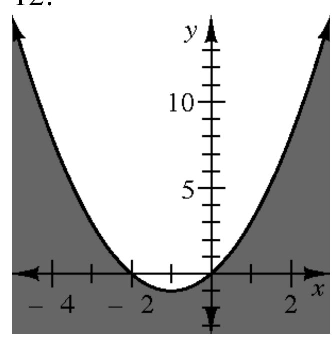

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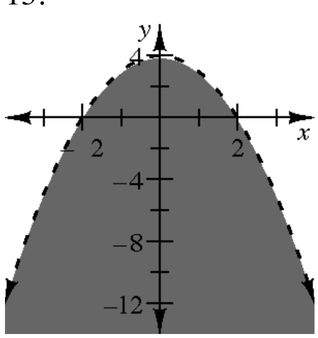

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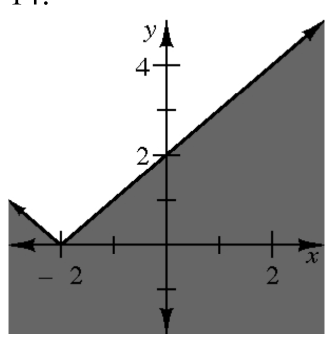

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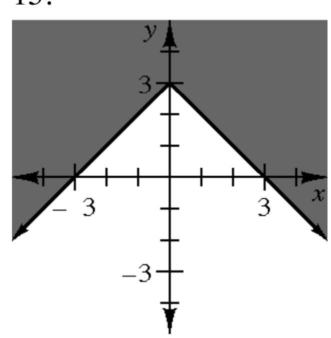


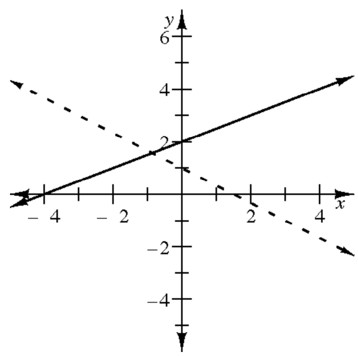
SYSTEMS OF INEQUALITIES 9.4.1 through 9.4.3

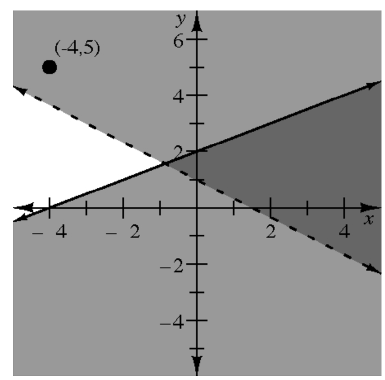
To **graph the solutions to a system of inequalities**, follow the same procedure outlined in the previous section but do it twice—once for each inequality. The solution to the system of inequalities is the **overlap** of the shading from the individual inequalities.

Example 1

Graph the solutions to the system $y \le \frac{1}{2}x + 2$ and $y > -\frac{2}{3}x + 1$.

Graph the lines $y = \frac{1}{2}x + 2$ and $y = -\frac{2}{3}x + 1$. The first is solid and the second is dashed. Test the point (–4, 5) in the first inequality.





$5 \le \frac{1}{2}(-4) + 2$, so $5 \le 0$

This inequality is false, so shade on the opposite side of the boundary line from (–4, 5), that is, below the line.

$5 > -\frac{2}{3}(-4) + 1$, so $5 > \frac{11}{3}$

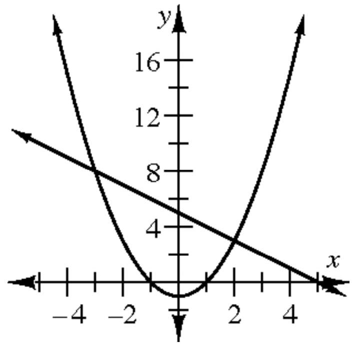
Test the same point in the second inequality. This inequality is true, so shade on the same side of the boundary line as (–4, 5), that is, above the line.

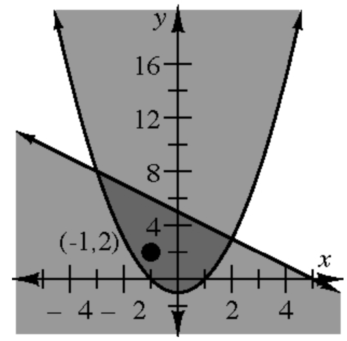
The solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the second graph above right.

Example 2

Graph the solutions to the system $y \le -x + 5$ and $y \ge x^2 - 1$.

Graph the line $y = -x + 5$ and the parabola $y = x^2 - 1$ with a solid line and curve.





$2 \le -(-1) + 5$, so $2 \le 6$

Test the point $(-1, 2)$ in the first inequality. This inequality is true, so shade on the same side of the boundary line as $(-1, 2)$, that is, below the line.

$2 \ge (-1)^2 - 1$, so $2 \ge 0$

Test the same point in the second inequality. This inequality is also true, so shade on the same side of the boundary curve as $(-1, 2)$, that is, inside the curve.

The solutions are in the overlap of the two shaded regions shown by the darkest shading in the second graph above right.

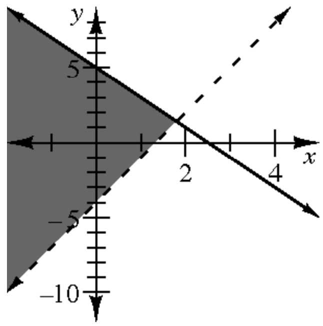
Problems

Graph the solutions to each of the following pairs of inequalities on the same set of axes.

1. $y > 3x - 4$ and $y \le -2x + 5$
2. $y \ge -3x - 6$ and $y > 4x - 4$
3. $y < -\frac{3}{5}x + 4$ and $y < \frac{1}{3}x + 3$
4. $y < -\frac{3}{7}x - 1$ and $y > \frac{4}{5}x + 1$
5. $y < 3$ and $y > \frac{1}{2}x + 2$
6. $x \le 3$ and $y < \frac{3}{4}x - 4$
7. $y \le 2x + 1$ and $y \ge x^2 - 4$
8. $y < -x + 5$ and $y \ge x^2 + 1$
9. $y < -x + 6$ and $y \ge |x - 2|$
10. $y < -x^2 + 5$ and $y \ge |x| - 1$

Answers

1.



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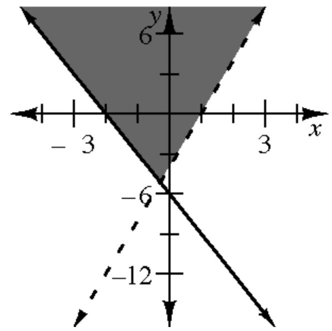

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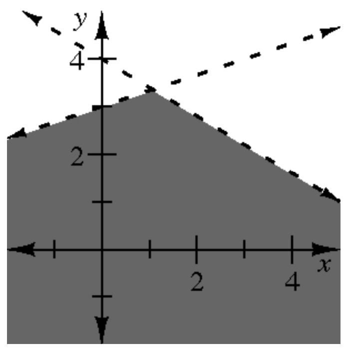

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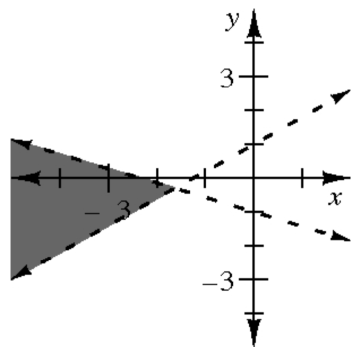

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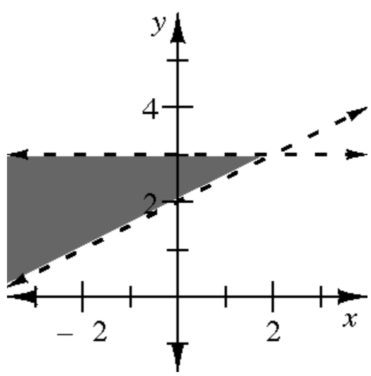

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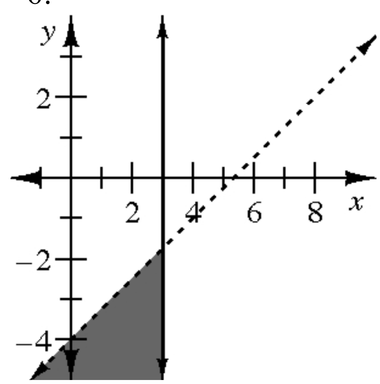

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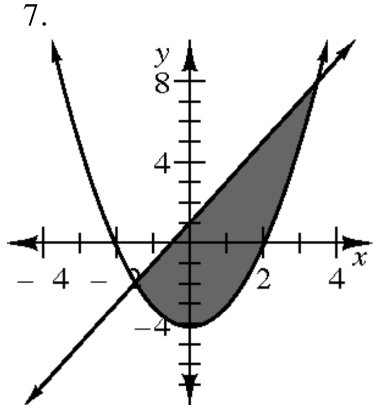

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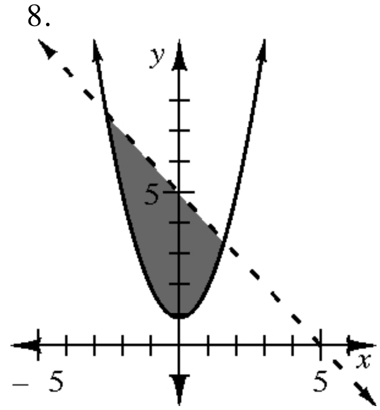
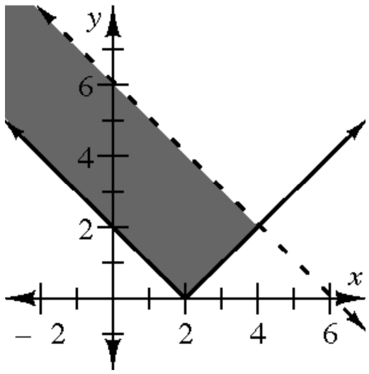


9.



10.